# Exam 1

Math 309 Fall 2016

October 5, 2016

No notes, calculators, or other electronic devices allowed.

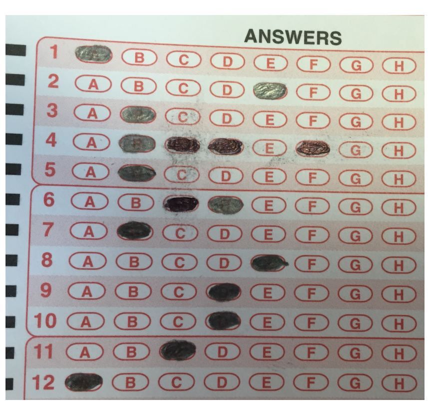
**Part I: Multiple Choice** Each problem in this section is worth five points.

1. Let C=AB where $A = \begin{bmatrix} 10 & 2 & 17 \\ 19 & 30 & 34 \\ 12 & 36 & 36 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & \frac{3}{4} & 1 \\ \frac{5}{6} & 2 & 4 \\ 1 & 3 & 0 \end{bmatrix}$. Find $c_{23}$.

   A) 139 B) 189 C) 0 D) $-12$ E) $\frac{10}{7}$ F) None of the above.

2. Let $A = \begin{bmatrix} 17 & 31 \\ -12 & 22 \end{bmatrix}$ and $B = \begin{bmatrix} 122 & \frac{31}{89} & 12 \\ 178 & 30 & 0 \end{bmatrix}$. Among the options below, find the one for which all of the listed expressions can be legally calculated applying the rules of matrix algebra to $A$ and $B$.

   A) $AB$, $BA$, and $A+B$
   B) $AB$ and $BA$
   C) $A+B$ and $A-B$
   D) $AB$ and $A+AB$
   E) $AB$ and $B+AB$



F) None of the above.
G) More than one of the above

3. Consider the system of equations

$$5x_1 - 7x_2 = 12$$

$$25x_1 + hx_2 = k.$$

Give conditions on $h$ and $k$ such that the system has *many* solutions.

A) $h = -35$ B) $h = -35$ and $k = 60$ C)$h \neq -35$ and $k = 60$
D)$h = -35$ and $k \neq 60$ E) $h \neq -35$ and $k \neq 60$ F) None of the above.

4. Which statement below is true?

   (a) Geometrically, the span of any two vectors may be seen as a plane.
   (b) The solution set of a linear system whose augmented matrix is $[\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{b}]$ is the same as the solution set to the vector equation $x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3} = \mathbf{b}$.
   (c) If a set of vectors are linearly dependent, then every vector in the set can be written as a linear combination of the other vectors in the set.
   (d) If a set of vectors are linearly dependent, then some vector in the set is a linear combination of another vector in the set.

A) a B) b C) d D) c E) e F) None of the above

5. If $T : P_1 \to P_1$ is a transformation on linear functions such that $T(1 + x) = 3 + 2x$ and $T(2 - 3x) = 11 - x$, find $T(3 + 2x)$.

A) 75-30x B) 10+5x C) 12-3x D) 12+42x E) 12x F) None of the above

6. Suppose $A$ is an $n \times n$ matrix. Which of the following statements are NOT equivalent to $A$ being invertible?

(a) If $T$ is a linear transformation and $A$ is the standard matrix for $T$, then $T$ is invertible.

(b) The solution to the system $Ax = 0$ has *no* non-trivial solutions.

(c) The columns of $A$ are linearly dependent.

(d) $A$ has $n$ pivots.

(e) The transformation $T(x) = Ax$ is a one-to-one map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

A) a  B) b  C) d  D) c  E) e  F) None of the above–all are equivalent.

7. Which of the following transformations are linear?

   (a) $T(x_1, x_2) = (\sin(x_1), \cos(x_2))$
   (b) $T(x_1, x_2, x_3) = (12x_1 + 6x_2, 10x_2 + 11x_3, x_1 + x_2 + 9x_3)$
   (c) $T(x_1, x_2, x_3) = (x_1^3, x_2^2, x_3)$
   (d) $T(x_1, x_2, x_3) = (x_1, x_2, \sqrt{x_3})$
   (e) $T(x_1, x_2, x_3) = (0, 1, 2)$

   A) a B) b C) c D) d E) e F) None of the above G) More than one of the above

8. Given a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, with associated matrix $A$, in which of the following situations is it *possible* that $T$ might be one-to-one?

   (a) The columns of $A$ are linearly dependent.
   (b) $m = n$, and $A^T$ is not an invertible matrix.
   (c) $m = n$, and the columns of $A$ do not span $\mathbb{R}^n$.
   (d) $n > m$
   (e) $n < m$

   A) a B) b C) c D) d E) e F) None of the above G) More than one of the above

9. For a certain linear transformation $T$ with associated matrix $A$, all of the following statements are true except for one. Assuming only one statement is false, which one is the false statement?

   (a) The transformation is one-to-one.
   (b) The equation $T(x) = 0$ has only the trivial solution.
   (c) $T : \mathbb{R}^n \to \mathbb{R}^m$, where $n < m$.
   (d) The columns of $A$ are linearly dependent

   A) a B) b C) c D) d E) Based on the information, they might all be true

10. Which statement about matrices below is not true?

(a) $(A^T)^T = A$
(b) $(A^{-1})^{-1} = A$
(c) $(A+B)^T = A^T + B^T$
(d) $(A+B)^{-1} = A^{-1} + B^{-1}$
(e) $(AB)^T = B^T A^T$
(f) $(AB)^{-1} = B^{-1}A^{-1}$

A) a B) b C) c D) d E) All are true F) More than one is false

11. Determine the number of free variables for the system

$$2x_1 + x_2 - 3x_3 + 13x_4 = 6$$

$$x_1 + 2x_2 + 3x_3 + 5x_4 = 12.$$

A) 0 B) 1 C) 2 D) 3 E) 4 F) None of the above

12. Consider a transformation $T : \mathbb{R}^r \to \mathbb{R}^s$, with associated matrix $A$ and $B$ the row-echelon reduction of $A$. If $T$ is onto, which of the following might be true?

(a) $r = 7$, $s = 5$ and $B$ has five pivots.
(b) $r = 5$, $s = 7$ and $B$ has five pivots.
(c) $r = 5$, $s = 7$ and $B$ has four pivots.
(d) $r = 7$, $s = 5$ and $B$ has four pivots.
(e) $T$ is not onto.

A) a B) b C) c D) d E) e F) None of the above

**Part II: Hand graded**

Show all appropriate work to receive full credit. Each question is worth 10 points.

1. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points around the origin by an angle of $\frac{\pi}{2}$ radians, then reflects points through the line $x_2 = x_1$.

   (a) Give the matrix $A_1$ for the transformation which rotates points around the origin by an angle of $\frac{\pi}{2}$ radians.

   (b) Give the matrix $A_2$ for the transformation which reflects points through the line $x_2 = x_1$.

   (c) Give the matrix $A_T$ for the T described above.

   (d) Describe the relation between $A_1$, $A_2$, and $A_T$, showing any appropriate calculations supporting your claim.

2. Clearly state whether each statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample or a reason why it cannot be true.

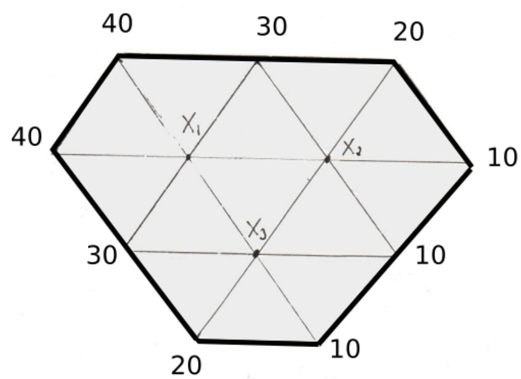
   (a) If $A$ and $B$ are square $(n \times n)$ matrices, then

$$AB = BA.$$

   (b) If $A$ and $B$ are invertible $n \times n$ matrices, then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

   (c) If $A$ is invertible, then $Ax = b$ has the unique solution $x = A^{-1}b$.




3. In the figure above temperatures are given at edge points for a non-regular hexagonal plate. (You may assume all of the interior triangles are equilateral.) Use a variation on the heat transfer model used in class to answer the following questions.

   (a) Set up a system of linear equations in **three** variables which gives the temperature at points $x_1$, $x_2$, and $x_3$.

   (b) Give the augmented matrix for your system.

   (c) Solve for $x_1$, $x_2$, and $x_3$, either by row reducing the system or by a shorter calculation taking advantage of any symmetries you might observe.

4. For each matrix below, either give a reason why it is not invertible or find the inverse.

(a) $A = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -3 & 17 \\ 4 & 20 & 6 \\ 3 & -9 & 51 \end{bmatrix}$

(c) $B = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$