Exam 2

Math 309

November 9, 2016

You may use a scientific calculator, but no notes, graphing calculators, or other electronic devices allowed. Fill out your scantron cards with your name, *including your official "preferred" name*, and your ID number.

Part I: Multiple Choice (30) Each problem in this section is worth five points.

1. If $A = \begin{bmatrix} -12 & 0 & 13 & 42 & 3 & -62 & 0 & 98 & 0 & 12 \\ 0 & -2 & 65 & 55 & 55 & 48 & 7 & 48 & 19 & 0 \\ 0 & 0 & 14 & 14 & 15 & 16 & -3 & 223 & 23 & 42 \\ 0 & 0 & 0 & 8 & 42 & 62 & 27 & -28 & 12 & -10 \\ 0 & 0 & 0 & 0 & -51 & -13 & 17 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 23 & 86 & -9 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & \pi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 101 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 \end{bmatrix}$, then

The diagonal has five negatives, so the product is negative.

A) det $A < 0$ B) det $A = 0$ C) det $A > 0$ D) det A is not a real number E) none of the above

2. If A and B are 3×3 matrices, $\det(A) = 3$ and $\det(B) = 2$, what is $\det((2A)B^{-1})$?

A) 1 B) -1 C) 4 D) 12 E) -12 F) 24 G) -24 H) none of the above

$= 2^3 \det A \cdot \frac{1}{\det B}$
$= 8 \cdot 3 \cdot \frac{1}{2}$

3. Let A be an $m \times n$ matrix such that there is one free variable in the system $A\mathbf{x} = \mathbf{0}$. What is the rank of A?

A) m B) n C) n - m D) m-1 E) n-1 F) n-m+1 G) m + n +1 H) none of the above

By the Rank Theorem $\text{rank}\,A + \underbrace{\dim(\text{Nul}\,A)}_{=\#\text{free var}} = n$

$a^2 = 4(a-1)$

4. Find all a such that $\det \begin{bmatrix} a & 0 & 2 & 4 \\ 0 & a & 0 & 0 \\ 0 & 1 & 4 & 5 \\ 0 & 3 & 3 & 4 \end{bmatrix} = \det \begin{bmatrix} 4 & 0 & 0 \\ -2 & a & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$a^2 - 4a + 4 = 0$

$(a-2)^2 = 0$

A) all real numbers a B) no real numbers a C) 0, -2 D) -2, 2
E) 2 F) 4 G) 2, 4 H) none of the above

Area 4π

5. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\mathbf{x}) = A\mathbf{x}$ for a certain matrix A. If S a circle of radius 2, the image under the transformation $T(S)$ is an ellipse with area 8π. Which of the following matrices might be A? Need $|\det A| = 2$

A) $\begin{bmatrix} 517 & 330 \\ -123 & 223 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$ D) $\begin{bmatrix} 2 & 0 \\ -12 & -3 \end{bmatrix}$

E) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ F) $\begin{bmatrix} 6 & 0 \\ -1 & 6 \end{bmatrix}$ G) $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ H) $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

I) None of the above

6. For an $m \times n$ matrix A, which of the following is not equal to the rest?

A) the rank of A B) the dimension of the column space of A C) the dimension of the row space of A D) the dimension of the null space of A
E) the number of pivot columns of A

Part II: True or False (30) Determine if each of the following statements is true or false. Each problem in this section is worth three points.

7. Any subspace of a vector space is also a vector space.

A) True B) False

8. The column space of a matrix A is the set of all linear combinations of columns of A.

A) True B) False

9. The column space of an $m \times n$ matrix A is the set of all $\mathbf{b} \in \mathbb{R}^{\mathbf{m}}$ such that there is a solution $x \in \mathbb{R}^n$ to the equation $A\mathbf{x} = \mathbf{b}$.

A) True B) False

10. For an $m \times n$ matrix A, the $Nul(A) = Col(A)$.

A) True B) False

11. By taking away vectors from the set $S = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\}$ we can obtain a basis for $\mathbb{R}^4$.

(Need to check that the given vectors span $\mathbb{R}^4$)

A) True B) False

12. A row space of a matrix A is a vector space.

A) True B) False

13. If $S = \{v_1, v_2, v_3, v_4\}$ is a linearly independent subset of $\mathbb{R}^4$, then S is a basis for $\mathbb{R}^4$.

A) True B) False

14. The number of *free variables* in the equation $A\mathbf{x} = \mathbf{b}$ equals the dimension Nul(A).

A) True B) False

15. The vectors $\begin{bmatrix} 1 \\ 10 \\ -1 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^5$.

A) True B) False

16. If $E_1, E_2, ..., E_n$ are elementary matrices and $E_1 \cdot E_2 \cdot ... \cdot E_n A = B$, then $\det A = \det B$.

A) True B) False

det E_i might not be one, if it is scaling or interchanging rows.

Part III: Short Answer(20)

You do not need to show any work for this part, just answer each question or fill in the blank.

(6) 17. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (in coordinates of the standard basis for $\mathbb{R}^2$) and let $\mathcal{B} = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\}$ be an alternate basis. Find $[\mathbf{x}]_{\mathcal{B}}$. $\vec{X} = P_{\mathcal{B}}[X]_{\mathcal{B}}$ $\quad P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $P_{\mathcal{B}}^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}[X]_{\mathcal{B}}$$

$$\frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = [X]_{\mathcal{B}} \longrightarrow \boxed{[X]_{\mathcal{B}} = \begin{bmatrix} 3/5 \\ 1/5 \end{bmatrix}}$$

$$\hookrightarrow \frac{1}{5}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(6) 18. Let $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and let $U = \text{span}\{u_1, u_2\}$, $V = \text{span}\{v_1, v_2\}$. Find a nonzero vector which is in the subspace $U \cap V$.

$\boxed{\text{Answer: } \vec{X} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ (or any scalar times } \vec{X})}$

Method 1: Reduce $A = [\vec{u}_1\ \vec{u}_2\ \vec{v}_1\ \vec{v}_2]$ to get the solution $\vec{w} = t\begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. In this case, notice that $\vec{w}$ is not your answer, but gives the weights, such as $\vec{X} = -2\vec{u}_1 + 1\cdot\vec{u}_2$ or $\vec{X} = 1\cdot\vec{v}_1 + 1\cdot\vec{v}_2$.

Method 2: The two subspaces are planes in $\mathbb{R}^3$, find a vector in the line intersecting them. (Cross products are useful.)

Method 3: Try to find $a, b, c, \& d$ so that $\vec{X} = a\vec{u}_1 + b\vec{u}_2 = c\vec{v}_1 + d\vec{v}_2$. (Notice that $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$ for $A = [\vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2]$.

(8) 19. The matrix $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(2 pts each part)

(a) rank(A)= 3

(See EX 2 p. 233)

Give a basis for:

(b) row(A)

$\mathcal{B} = \{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$

Alternatively, rows 1, 2, 4 or 1, 3, 4 or 2, 3, 4 of A work, but notice rows 1, 2, 3 are dependent and give a smaller subspace.

(c) Col(A) $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

(Note that columns of the reduced matrix do not give a basis.)

(d) Nul(A) $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

Part IV: Longer Answer(20)

20. Clearly state whether each statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample or a reason why it cannot be true.

    (a) If A is a 2×2 matrix, then the determinant of A is equal to the determinant of its transpose. That is,
    $$det(A^T) = det(A).$$

True: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A^T) = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb = ad - bc = \det A$

    (b) If A is an $n \times n$ matrix, then the determinant of A is equal to the determinant of its inverse. That is,
    $$det(A^{-1}) = det(A).$$

False: Counterexample: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det A = 2$, while $\det A^{-1} = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$.

    (c) If H and K are subspaces of a vector space V, then the intersection $H \cap K$ is also a subspace.

True: Proof: (a) Show $\vec{0} \in H \cap K$

H + K are subspaces $\Rightarrow \vec{0} \in H$ and $\vec{0} \in K \Rightarrow \vec{0} \in H \cap K$

(b) Show $H \cap K$ is closed under addition

If $\vec{u}, \vec{v} \in H \cap K$, then $\vec{u} \in H$ + $\vec{u} \in K$ + $\vec{v} \in H, \vec{v} \in K$ $\Rightarrow \vec{u}+\vec{v} \in H$ + $\vec{u}+\vec{v} \in K \Rightarrow \vec{u}+\vec{v} \in H \cap K$

(c) Show closed under scalar multiplication

If k is a scalar + $\vec{u} \in H \cap K$, then $\vec{u} \in H \Rightarrow r\vec{u} \in H$ + $\vec{u} \in K \Rightarrow r\vec{u} \in K$ $\Rightarrow r\vec{u} \in H \cap K$

Therefore $H \cap K$ is a subspace of V.

21. For each matrix below, find the determinant of A.

(a) $A = \begin{bmatrix} 12 & 4 \\ -1 & 2 \end{bmatrix} = 28$

(b) $A = \begin{bmatrix} 10 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 \\ 4 & -9 & 5 & 5 \\ 4 & -9 & 5 & -2 \end{bmatrix} = -700$

(c) $A = \begin{bmatrix} \frac{1}{2} & \frac{5}{8} & \frac{1}{3} \\ 12 & 15 & 8 \\ \frac{1}{30} & 32 & 13 \end{bmatrix} = 0$ (Notice row one $= \frac{1}{24}$ row two, So it must be zero.)