Part IV: Longer Answer(20)

Show all appropriate work to receive full credit.

18. Clearly state whether each statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample or a reason why it cannot be true.

(a) If A and B are square $(n \times n)$ matrices, then

$$det(AB) = det((AB)^{-1}).$$

False $\det(I \cdot 2I) = 4$ but $(I \cdot 2I)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$, $\det\left(\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\right) = \frac{1}{4}$

(Most counterexamples will demonstrate)

(b) Let $\mathcal{P}_2$ be the vector space of quadratic polynomials $p(t)$.
Then $H = \{p(t) \in \mathcal{P}_2 | p(t) = at^2 + b, a, b \in \mathbb{R}\}$ is a subspace.

True: Proof: By the definition of a subspace, need to show

① $0 \in H$: Let $a=b=0$, $p(t)=0 \in H$

② Closed under addition

Let $p_1 = a_1 t^2 + b_1$, $p_2 = a_2 t^2 + b_2 \in H$.

Then $p_1 + p_2 = (a_1+a_2)t^2 + (b_1+b_2)$, $a_1+a_2, b_1+b_2 \in \mathbb{R}$

③ Closed under scalar multiplication

$kp(t) = kat^2 + kb$, $ka, kb \in \mathbb{R}$, so $kp(t) \in H$.

(c) The map $D : M_{n\times n} \to \mathbb{R}$ defined by $D(A) = det(A)$ is a linear map.

False $D(kA) = k^n \det A \neq k \det A$, so D fails the scalar multiplication of linearity.

(Note that D is multilinear, which is distinct from linear.)