Note: *On a Hardy Space Approach to the RH*

Christopher Felder
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✉ *cfelder@wustl.edu*
math.wustl.edu/∼cfelder

Approximation

Somewhat recently, following the work Nyman [5], Beurling [3], and Báez-Duarte [1], S. Waleed Noor proved the following result ([4], Theorem 6):

> For $j \geq 2$, let
> $$h_j = \frac{1}{1-z} \log\left(\frac{1-z^j}{j(1-z)}\right).$$
> The Riemann hyposthesis is true if and only if the constant function 1 is in the closed linear span of $\{h_j : j \geq 2\}$ in H^2.

Of course, here, H^2 is the set of analytic functions on the unit disk $\mathbb{D}$ with square-summable Maclaurin coefficients. One way to approach this problem, in the spirit of least-squares (e.g. see [2]), is to notice that 1 is in the closed linear space of $\{h_j : j \geq 2\}$ if and only if we can find constants $(c_{j,N})_{j=2}^N$ such that $\left\|\sum_{j=2}^N c_{j,N} h_j - 1\right\|_{H^2} \to 0$ as $N \to \infty$. Asking this norm to decay *optimally* gives a minimization problem for each N:

$$\min_{c_2,\ldots,c_N \in \mathbb{C}} \left\|\sum_{j=2}^N c_j h_j - 1\right\|_{H^2}.$$

Given the Hilbert space structure, we see that this minimization is really just asking us to find, for each $N \geq 2$, the orthogonal projection of 1 onto the subspace $V_N := \text{span}\{h_j : j = 2, \ldots, N\}$. Denoting the orthogonal projection of 1 onto V_N as $P_N 1$, and noting then that $1 - P_N 1$ is orthogonal to V_N, we see that the corresponding coefficients of each h_j in $P_N 1$, say $c^*_{j,N}$, must satisfy the orthogonality condition

$$1 - \sum_{j=2}^N c^*_{j,N} h_j \perp \sum_{n=2}^N c_n h_n$$

for all constants $c_2, \ldots, c_N \in \mathbb{C}$. Namely, taking $\sum_{n=2}^N c_n h_n = h_k$ for each $k = 2, \ldots N$, we get

$$\left\langle 1 - \sum_{j=2}^N c^*_{j,N} h_j, h_k \right\rangle_{H^2} = 0.$$

Moving some things around, we have

$$\sum_{j=2}^N c^*_{j,N} \langle h_j, h_k\rangle_{H^2} = \langle 1, h_k\rangle_{H^2}.$$

Putting this together for each $k = 2, \ldots, N$, we arrive at the following system:

$$(\langle h_j, h_k \rangle_{H^2})_{2 \le j,k \le N} \left(c^*_{2,N}, \ldots, c^*_{N,N} \right)^T = (\log(1/2), \ldots, \log(1/N))^T$$

where the right hand side follows from noting that 1 is the reproducing kernel at 0 for H^2 and $\overline{h_k(0)} = \log(1/k)$. Further, we see that square distance between 1 and V_N is

$$\begin{aligned}
\mathsf{dist}^2(1, V_N) :&= \|P_N 1 - 1\|^2_{H^2} \\
&= \|P_N 1\|^2 - 2\Re\langle P_N 1, 1\rangle_{H^2} + \|1\|^2 \\
&= 1 - \|P_N 1\|^2_{H^2} \\
&= 1 - \sum_{j=2}^{N} c^*_{j,N} h_j(0)
\end{aligned}$$

where the last and next to last equalities hold since $\|P_N 1\|^2 = \langle P_N 1, P_N 1, \rangle = \langle P_N 1, 1 \rangle = (P_N 1)(0)$.
This leads us to an additional equivalence:

> The Riemann hypothesis is true if and only if $1 - \sum_{n=2}^{N} c^*_{n,N} \log(1/n) \to 0$ as $N \to \infty$, where, for each $N \ge 2$, $c^*_{2,N}, \ldots, c^*_{N,N} \in \mathbb{C}$ is the solution to the linear system
>
> $$(\langle h_j, h_k \rangle_{H^2})_{2 \le j,k \le N} \left(c^*_{2,N}, \ldots, c^*_{N,N} \right)^T = (\log(1/2), \ldots, \log(1/N))^T .$$

So why not just compute the solution to the system above for general N, take a limit, and collect the million dollars? Well, the first trouble here comes not in inverting the matrices G_N, which will be challenging in its own right, but in calculating the inner products $\langle h_j, h_k \rangle_{H^2}$.
This approach is not the first in taking up least-squares against the RH. Most recently, in [2], Bellemare, Langlois, and Ransford confront a similar problem in $L^2(0,1)$.
For each integer $k \ge 2$, define $f_k : (0,1] \to \mathbb{R}$ by

$$f_k(x) := \frac{1}{k}\left[\frac{1}{x}\right] - \left[\frac{1}{kx}\right]$$

where $[y]$ is the integer part of $y \in \mathbb{R}$. Via theorems of Nyman and Báez-Duarte, they relay the following:

> Let d_n be the distance between 1 and the span of $\{f_2, \ldots, f_n\}$ in $L^2(0,1)$.
>
> The Riemann hypothesis is true if and only if $d_n \to 0$ as $n \to \infty$.

They then provide some concise and useful results surrounding this problem and produce a quite robust numerical exploration, leaving with a conjecture about the positivity of a certain matrix arising from the minimization problem associated to the above distance problem.

A Short Numerical Experiment

Noor (and an anonymous referee) provide a Maclaurin series representation of h_k ([4], pp. 249) as

$$h_k(z) = \sum_{n\geq 0}\left(\log(1/k) + \sum_{j=1}^{n}\frac{1}{j}(1-k[k|j])\right) z^n$$

where $[k|j]$ equals 1 if k divides j and 0 otherwise. This gives

$$\langle h_j, h_k\rangle_{H^2} = \sum_{n\geq 0}\left(\log(1/j) + \sum_{m=1}^{n}\frac{1}{m}(1-j[j|m])\right)\left(\log(1/k) + \sum_{m=1}^{n}\frac{1}{m}(1-k[k|m])\right).$$

Suppose we let

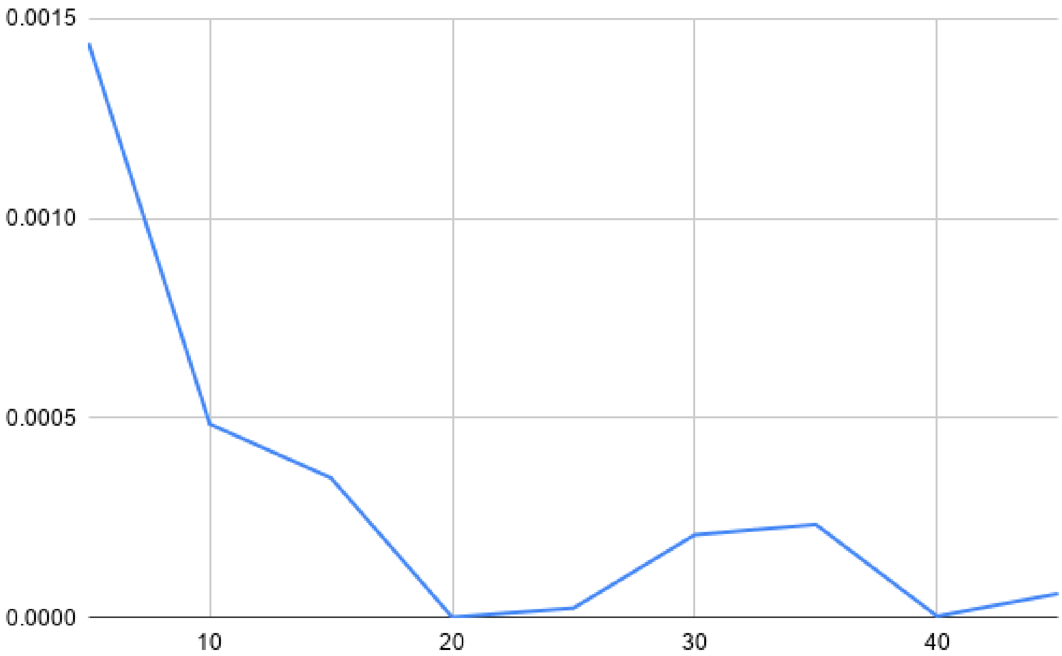
$$g_{j,k}^{N} := \sum_{n=0}^{N}\left(\log(1/j) + \sum_{m=1}^{n}\frac{1}{m}(1-j[j|m])\right)\left(\log(1/k) + \sum_{m=1}^{n}\frac{1}{m}(1-k[k|m])\right).$$

be the N-th truncation of $\langle h_j, h_k\rangle_{H^2}$. Then a truncated reformulation of the H^2 minimization problem is as follows:

> The Riemann hypothesis is true if and only if $1-\sum_{n=2}^{N} a_{n,N}\log(1/n) \to 0$ as $N\to\infty$, where, for each $N\geq 2$, $a_{2,N},\ldots,a_{N,N}\in\mathbb{C}$ is the solution to the linear system
>
> $$\left(g_{j,k}^{N}\right)_{2\leq j,k\leq N}(a_{2,N},\ldots,a_{N,N})^{T} = (\log(1/2),\ldots,\log(1/N))^{T}.$$

A priori, the matrix $\left(g_{j,k}^{N}\right)_{2\leq j,k\leq N}$ is not invertible, so the solution to the system does not necessarily uniquely exist. However, as will shortly be evident, the matrix of truncated inner products above is invertible for some small N. It should also be noted that the matrix $\left(g_{j,k}^{N}\right)_{j,k}$ could be of any size and does not have to coincide with the length of the truncation of $\langle h_j, h_k\rangle_{H^2}$, it is simply stated as such for numerical ease. Numerical calculations give the following values for N versus $\delta_N^2 := 1-\sum_{j=2}^{N} a_{j,N}h_j(0)$:

N	δ_N^2
5	0.001439242733910
10	0.000484875828723
15	0.000349984890394
20	0.000001209978708
25	0.000023766812139
30	0.000207317697316
35	0.000233307885123
40	0.000003705806710
45	0.000060039097384



N versus δ_N^2 for $N = 5, 10, \ldots, 45$

Note that the values tangential to the horizontal axis are not actually zero (only appearing so due to scaling). These computations show, for small N, that δ_N^2 is not monotonic. Of course, this observation has no bearing on the overall question at hand.
Nonetheless, it is not the point to become so overzealous or naive to try to tackle this problem at the waist, but, instead, to be motivated by this equivalence to find some tractable work on its peripheries.

References

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