Math 310 Class Notes 2: The Birth of Natural Numbers

Definition 1. *A set S is called an inductive set if the empty set $\phi \in S$ and if a set $a \in S$ then its successor $a' := a \cup \{a\} \in S$.*

For instance, the set A in Axiom 8 of set theory is an inductive set; let $P(A)$ denotes the power set of A, whose existence is warranted by Axiom 6 of set theory. The collection of all inductive subsets of A forms a set $\mathcal{F}$. This is because

$$\mathcal{F} = \{C \in P(A) : C \text{ is an inductive set}\}$$

and the right hand side is a set by Axiom 2.

Definition 2. $\mathbb{N} := \cap_{C \in \mathcal{F}} C = \{x \in A : x \in C \text{ for all } C \in \mathcal{F}\}$ *called the set of natural numbers.*

Question. Why is $\mathbb{N}$ a set? Why is it an inductive set?

If you can answer the above question right, then you can deduce that $\mathbb{N}$ is the *smallest* inductive subset of A, in the sense that $\mathbb{N}$ is contained in every inductive subset of A. (WHY?)

However, Axiom 8 does not exclude the possibility that there may exist another inductive set B, In which case we can follow the above definition to collect all the inductive subsets D of B and take the intersection of all these D to come up with a set, say, $\mathbb{N}_1$. If $\mathbb{N} \neq \mathbb{N}_1$, then our definition of the set of natural numbers would be flawed, as it would depend on the inductive set given in Axiom 8; different inductive sets in Axiom 8 might give different sets of natural numbers. Fortunately, this will not happen as the following theorem demonstrates.

Theorem 3. $\mathbb{N}$ *is well-defined independent of A.*

Proof. We keep the preceding notations. Since $\mathbb{N}$ and $\mathbb{N}_1$ are both inductive sets, their intersection must be an inductive set as well. But now $\mathbb{N} \cap \mathbb{N}_1 \subset \mathbb{N} \subset A$ is an inductive subset of A, and so $\mathbb{N}$ is contained in $\mathbb{N} \cap \mathbb{N}_1$, because $\mathbb{N}$ is the smallest inductive subset of A. Therefore, $\mathbb{N} \subset \mathbb{N}_1$. (WHY?) Likewise, $\mathbb{N}_1 \subset \mathbb{N}$. Hence $\mathbb{N} = \mathbb{N}_1$. □

The starting point of constructing real numbers and their algebraic operations is the collection of five axioms characterizing the set of natural numbers, called the Axioms of Peano, who introduced these axioms in 1889 predating the axioms of set theory established in 1908-1922.

Theorem 4. *(Peano's Axioms, Set-theoretic Version)* $\mathbb{N}$ *satisfies the following properties.*

I: $\phi \in \mathbb{N}$.

II: *If* $n \in \mathbb{N}$*, then* $n' \in \mathbb{N}$*.*
III: *If* $m, n \in \mathbb{N}$ *and if* $m' = n'$*, then* $m = n$*.*
IV: *If* $n \in \mathbb{N}$*, then* $n' \neq \phi$*.*
V: *Let* $S \subset \mathbb{N}$*. Suppsoe* $\phi \in S$*. If* $n \in S$ *implies* $n' \in S$*, then* $S = \mathbb{N}$*.*

Question. Why is this theorem true?

If we agree to call the empty set 1. Then we arrive at the original axioms of Peano.

(**Peano's Axioms, Original Version, 1889**)

I: 1 **is a natural number.**
II: For every natural number n**, there exists one and only one natural number, denoted** n' **and called the successor of** n**.**
III: If m **and** n **are natural numbers and if** $m' = n'$**, then** $m = n$**. (No two different natural numbers have the same successors.)**
IV: For any natural number n**, we have** $n' \neq 1$**. (No natural number has** 1 **as a successor.)**
V: (Mathematical Induction) Let S **be a subset of natural numbers. Suppose** $1 \in S$**. If** $n \in S$ **implies** $n' \in S$**, then** S **contains all the natural numbers.**