CLASSIFICATION OF ISOPARAMETRIC HYPERSURFACES

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1. Early History of Isoparametric Hypersurfaces

Wikipedia. In physics, a *wavefront* is the locus of points having the same phase: a line or curve in 2-d or a surface for a wave propagating in 3-d.

A typical example is the crests of ocean waves forming wave fronts. A skillful surfer, on the other hand, knows how to ride a wavefront below the crest.

Question 1. (Laura, 1918 [16])*: What are the wavefronts whose front speed remains constant on each front surface?*

The wave equation is

$$\Delta\phi = \frac{\partial^2\phi}{\partial t^2}$$

Wave fronts are *level surfaces* of ϕ, at each moment, which propagate along the normal directions of the level surfaces. That the front speed remains constant on each level surface means

$$|\nabla\phi| = \text{ change per unit length of } \phi \text{ along the normal} = a(\phi),$$
$$ds/dt = b(\phi),$$

for some smooth functions a and b, where s is the distance a wavefront travels. Therefore,

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial s}\frac{ds}{dt} = a(\phi)b(\phi) := c(\phi),$$
$$\Delta\phi = \frac{\partial^2\phi}{\partial t^2} = c'(\phi)c(\phi).$$

Definition 2. *A smooth function f over $\mathbb{R}^3$ is transnormal if*

$$|\nabla f| = A(f)$$

for some smooth functions A. A transnormal function is isoparametric if

$$\Delta f = B(f).$$

Let c be a regular value of an isoparametric function f. The level surface $f^{-1}(c)$ is called an isoparametric surface.

Theorem 3. (Somigliana, 1918-1919 [28]) *A transnormal function f is isoparametric if and only if each regular level surface of f has constant mean curvature.*

Theorem 4. (Somigliana) *The regular level surfaces of an isoparametric function must be either all spheres, all cylinders or all planes.*

This theorem was rediscovered later by Segre [26] in 1924 and Levi-Civita in 1937. Levi-Civita's approach lent its way to the subsequent generalization to higher dimensions.

Lemma 5. (Levi-Civita, 1937 [17]) *A transnormal f is isoparametric if and only if the two principal curvatures of each regular level surface are constant.*

Theorem 6. (Segre, 1938 [27]) *The same conclusion of Somigliana holds on $\mathbb{R}^n$. That is, an isoparametric hypersurface, which is a regular level hypersurface of an isoparametric function f over $\mathbb{R}^n$ satisfying*

$$|\nabla f| = A(f), \quad \Delta f = B(f),$$

is either a hypersphere, a hyperplane, both are totally umbilic (one principal curvature), or a cylinder $S^k \times \mathbb{R}^{n-1-k}$.

Theorem 7. (Cartan, 1938 [3]) *The same conclusion of Segre holds on the hyperbolic space H^n of constant curvature -1. That is, an isoparametric hypersurface in H^n must be either a sphere, a hyperbolic H^{n-1}, a Euclidean $\mathbb{R}^{n-1}$ (called a horosphere), all three are totally umbilic, or a cylinder $S^k \times H^{n-k-1}$.*

IDEA: Cartan first showed that a transnormal function f (in the standard space forms) is isoparametric if and only if each regular level hypersurface of f has *constant mean curvature*, if and only if the shape operator of each regular level hypersurface has *constant principal curvatures* (*values*), counting multiplicities.

In the hyperbolic case, he then showed again that there are at most two principal values of the shape operator. Indeed, let $\lambda_1, \cdots, \lambda_{n-1}$ be the principal values of an isoparametric hypersurface in a standard space form of dimension n with constant curvature C. Then we have

$$(1) \qquad \sum_{j \neq k} m_j \frac{C + \lambda_k \lambda_j}{\lambda_k - \lambda_j} = 0, \quad \text{summed on } j,$$

referred to by Cartan as the "Fundamental Formula", which was proved by Segre in the Euclidean case and by Cartan in general. Here, m_j is the multiplicity of λ_j and $\lambda_i \neq \lambda_j$ if $i \neq j$.

Remark 8. *Say,* $C = 1$*. That is, the ambient space is the unit sphere in which the isoparametric hypersurface sits. Write*

$$\lambda_j = \cot(\theta_j).$$

Then the fundamental formula is nothing but

$$\sum_{j \neq k} \cot(\theta_k - \theta_j) = 0,$$

which carries a significant geometric meaning. Namely, in the spherical case, wavefronts, that is, the 1-parameter family of isoparametric hypersurfaces, eventually degenerate to two subspaces of much smaller dimensions whose mean curvatures are zero.

The case $C = 1$ is remarkably deep! At this point, 95 years after Laura first investigated isoparametric surfaces, there remains the last case (out of infinitely many) to be classified:

Classify the isoparametric hypersurfaces in S^{31} *with four principal values of multiplicities* $7, 7, 8, 8$.

Different fields of mathematics, such as differential geometry, algebraic geometry, algebraic topology, homotopy theory, K-theory, representation theory, etc., interplay in this arena.

Definition 9. g *is the number of principal values of an isoparametric hypersurface in* S^n.

Theorem 10. (Cartan, 1939-1940 [4], [6])

$g = 1$. *This is the* 1*-parameter family of parallel hyperspheres degenerating to the North and South Poles, called the focal submanifolds of the family.*

$g = 2$. *This is the* 1*-parameter family of generalized tori* $S^k \times S^{n-k-1}$*, whose points are*

$$(x_0, \cdots, x_k, x_{k+1}, \cdots, x_n),\ x_0^2 + \cdots + x_k^2 = r^2,\ x_{k+1}^2 + \cdots + x_n^2 = s^2,\ r^2 + s^2 = 1,$$

which degenerates to two focal submanifolds S^k *and* S^{n-k-1} *of radius* 1 *as* r *approaches* 0 *or* 1.

$g = 3$. I. *The three principal values have equal multiplicity* $m = 1, 2, 4,$ *or* 8.

II. *In the ambient Euclidean space* $\mathbb{R}^{n+1} \supset S^n$*, there is a homogeneous polynomial* F *of degree* 3*, satisfying*

$|\nabla F|^2 = 9r^2$, r is the Euclidean radial distance, and,

$\Delta F = 0,$

whose restriction to S^n is exactly the isoparametric function f. The range of f is $[-1,1]$ and ± 1 are the only critical values. Thus $f^{-1}(c), -1 < c < 1$, form a 1-parameter family of isoparametric hypersurfaces that degenerates to the two focal submanifolds $f^{-1}(1)$ and $f^{-1}(-1)$.

III. *The two focal submanifolds are the real, complex, quaternionic, or octonion projective plane corresponding to the principal multiplicity $m = 1, 2, 4,$ or 8. Each isoparametric hypersurface in the family is a tube around the projective plane.*

IV. *Let $\mathbb{F}$ be one of the normed algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. Let $X, Y, Z \in \mathbb{F}$ and $a, b \in \mathbb{R}$. Then*

$$\begin{aligned} F &= a^3 - 3ab^2 \\ &+ \frac{3a}{2}(X\overline{X} + Y\overline{Y} - 2Z\overline{Z}) \\ &+ \frac{3\sqrt{3}b}{2}(X\overline{X} - Y\overline{Y}) \\ &+ \frac{3\sqrt{3}}{2}((XY)Z + \overline{(XY)Z}) \end{aligned} \tag{2}$$

$g = 4$. *He assumed equal multiplicity m and classified the cases when $m = 1$ or 2.*

Question 11. (Cartan, 1940 [6])

(i) *What are the possible g?*
(ii) *Is equal multiplicity of principal values always true?*
(iii) *Are all isoparametric hypersurfaces homogeneous?*

Technically, it is more convenient to work with the following definition of an isoparametric hypersurface, though it is equivalent to the above one, as we will see later.

Definition 12. *A hypersurface in $\mathbb{R}^n, S^n$ or H^n, is called isoparametric if its principal values are everywhere constant, counting multiplicities.*

Question 13. *Classify all isoparametric hypersurfaces in spheres.*

This is Problem 34 on Yau's list of important open problems in geometry proposed in 1992 [32].

2. Development in the early 1970s, the homogeneous case

Nomizu wrote two papers [23], [24] in the early 1970s that revived the interest in isoparametric hypersurfaces. At about the same time, Takagi and Takahashi [30] classified homogeneous isoparametric hypersurfaces in spheres. Takagi and Takahashi's work was based on the comprehensive work of Cartan on the classification of symmetric spaces, and Hsiang and Lawson's 1971 work on the classification of homogeneous hypersurfaces in spheres.

Definition 14. *A connected hypersurface M in a smooth manifold X is called homogeneous if $I(X, M)$, the group of isometries of X leaving M invariant, acts transitively on M.*

It is clear that for such a hypersurface, the principal values of its shape operator are everywhere constant, counting multiplicities.

Theorems 6 and 7 classify all isoparametric hypersurfaces in $\mathbb{R}^n$ and H^n to be exactly the homogeneous hypersurfaces in these space forms.

What is interesting is then the spherical case.

We start with understanding the homogeneous ones:

Theorem 15. (Hsiang and Lawson, 1971 [15]) *A homogeneous hypersurface in a sphere is exactly a principal orbit of the isotropic representation of a rank 2 symmetric space.*

The following table is the collection of all symmetric spaces G/K of Types I and II whose isotropy representations give homogeneous (isoparametric) hypersurfaces M. There are at most two multiplicities $(m_1, m_2), m_1 \leq m_2$, for the g principal curvatures.

G	K	$\dim M$	g	(m_1, m_2)
$S^1 \times SO(n+1)$	$SO(n)$	n	1	$(1,1)$
$SO(p+1) \times SO(n+1-p)$	$SO(p) \times SO(n-p)$	n	2	$(p, n-p)$
$SU(3)$	$SO(3)$	3	3	$(1,1)$
$SU(3) \times SU(3)$	$SU(3)$	6	3	$(2,2)$
$SU(6)$	$Sp(3)$	12	3	$(4,4)$
E_6	F_4	24	3	$(8,8)$
$SO(5) \times SO(5)$	$SO(5)$	8	4	$(2,2)$
$SO(10)$	$U(5)$	18	4	$(4,5)$
$SO(m+2), m \geq 3$	$SO(m) \times SO(2)$	$2m-2$	4	$(1, m-2)$
$SU(m+2), m \geq 2$	$S(U(m) \times U(2))$	$4m-2$	4	$(2, 2m-2)$
$Sp(m+2), m \geq 2$	$Sp(m) \times Sp(2)$	$8m-2$	4	$(4, 4m-5)$
E_6	$(Spin(10) \times SO(2))/\mathbb{Z}_4$	30	4	$(6,9)$
G_2	$SO(4)$	6	6	$(1,1)$
$G_2 \times G_2$	G_2	12	6	$(2,2)$

3. Development in the early 1970s, the general case

Münzner [22] in 1973 proved a remarkable result that extended Cartan's investigation, recorded in Theorem 10, in a far-reaching manner:

Theorem 16. *Let M be any isoparametric hypersurfaces with g principal curvatures in S^n. Then we have the following.*

(1) *There is a homogeneous polynomial F, called* Cartan-Münzner *polynomial, of degree g over $\mathbb{R}^n$ satisfying*

$$|\nabla F|^2 = g^2 r^{2g-2}, \quad \Delta F = \frac{m_- - m_+}{2} g^2 r^{g-2},$$

where r is the radial function over $\mathbb{R}^{n+1}$.

(2) *Let $f := F|_{S^n}$. Then the range of f is $[-1, 1]$. The only critical values of f are ± 1. Moreover, $M_\pm := f^{-1}(\pm 1)$ are connected submanifolds of codimension $m_\pm + 1$ in S^n, called focal submanifolds, whose principal curvatures are $\cot(k\pi/g), 1 \leq k \leq g-1$.*

(3) *For any $c \in (-1, 1)$, $M_c := f^{-1}(c)$ is an isoparametric hypersurface with at most two multiplicities $m_\pm$ associated with the principal curvatures. In fact, if we order the principal curvatures $\lambda_1 > \cdots > \lambda_g$ with multiplicities $m_1, \cdots, m_g$, then $m_i = m_{i+2}$ with index modulo g; in particular, all multiplicities are equal when g is odd, and when g is even, there are at most two multiplicities precisely equal to $m_\pm$.*

(4) *Each of the 1-parameter isoparametric hypersurfaces is a tube around the two focal submanifolds, so that S^n is obtained by gluing two disk bundles over $M_\pm$ along the isoparametric hypersurface $M_0 := f^{-1}(0)$. As a consequence, algebraic topology implies that the only possible values of g are $1, 2, 3, 4,$ or 6.*

Corollary 17. *$M_\pm$ are minimal submanifolds of S^n. The minimality condition is exactly equation (1), the fundamental formula of Segre and Cartan.*

Corollary 18. *There is a unique minimal isoparametric hypersurface in the 1-parameter family M_c.*

Now that S^n is obtained by gluing two disk bundles over the focal submanifolds $M_\pm$ along an isoparametric hypersurface M, Münzner used algebraic topology to express the cohomology ring of M, with $\mathbb{Z}_2$ coefficients, as modules of those of $M_\pm$, whose module structures then give $g = 1, 2, 3, 4,$ or 6.

Based on Münzner's work, Ozeki and Takeuchi [25] in 1975-76 constructed two classes, each with infinitely many members, of *inhomogeneous* isoparametric hypersurfaces with $g = 4$. They also classified all isoparametric hypersurfaces with $g = 4$ when one of the multiplicities $m_\pm$ is 2, which are all homogeneous.

An important ingredient in their work is their expansion formula of the Cartan-Münzner polynomial:

$$F(tx + y + w) = t^4 + (2|y|^2 - 6|w|^2)t^2 + 8(\sum_{a=0}^{m_+} p_a w_a)t$$
$$+ |y|^4 - 6|y|^2|w|^2 + |w|^4 - 2\sum_{a=0}^{m_+}(p_a)^2 + 8\sum_{a=0}^{m_+} q_a w_a$$
$$+ 2\sum_{a,b=0}^{m_+}\langle \nabla p_a, \nabla p_b\rangle w_a w_b.$$

Here, x is a point on M_+, y is tangent to M_+ at x, and w is normal to M_+ with coordinates w_i with respect to the chosen orthonormal normal basis $\mathbf{n}_0, \mathbf{n}_1, \cdots, \mathbf{n}_{m_+}$ at x. Moreover, $p_a(y)$ (resp., $q_a(y)$) is the a-th component of the 2nd (resp., 3rd) fundamental form of M_+ at x. Furthermore, p_a and q_a are subject to ten convoluted equations, to be seen later, of which the first three assert that, since $S_{\mathbf{n}}$, the 2nd fundamental matrix of M_+ in any unit normal direction $\mathbf{n}$, has eigenvalues $1, -1, 0$ with fixed multiplicities, it must be that $(S_{\mathbf{n}})^3 = S_{\mathbf{n}}$.

The expansion formula coupled with the ten identities are fundamentally important for the classification of isoparametric hypersurfaces with $g = 4$.

4. Development in the 1980s

The multiplicities of the principal values for $g = 4$ and $g = 6$ had remained undetermined until Abresch [2] in 1983 extended Münzner's work to show, by algebraic topology, that for $g = 6$ we have $m_+ = m_- = 1$ or 2. This is in agreement with the multiplicities of the homogeneous examples. Although he derived some constraints in the case $g = 4$, among which we have, for instance, $m_+ = m_-$ implies $m_+ = m_- = 1$ or 2, etc., the case remained open.

Meanwhile, Ferus, Karcher and Münzner [14] in 1981 generalized the inhomogeneous examples of Ozeki and Takeuchi to construct infinitely many classes, each with infinitely many members, of *inhomogeneous* isoparametric hypersurfaces with $g = 4$. Their construction can be best motivated by the example in Nomizu's 1973 paper mentioned earlier:

Consider $\mathbb{C}^k = \mathbb{R}^k \oplus \mathbb{R}^k$ and write $z \in \mathbb{C}^k$ as $z = x + \sqrt{-1}y$ accordingly. Define an homogeneous polynomial of degree 4 on $\mathbb{C}^k$ by

$$\tilde{F} = (|x|^2 - |y|^2)^2 + 4(\langle x, y\rangle)^2.$$

Then F is an isoparametric function with multiplicities $\{1, k-2\}$. In fact, the isoparametric hypersurfaces are the principal isotropy orbits of the symmetric spaces $SO(k+2)/S(2) \times SO(k)$.

Note that $\tilde{f} = \tilde{F}|_{S^{2k-1}}$ has range $[0, 1]$. So we normalize it by defining $f := 1 - 2\tilde{f}$, or rather, by setting

$$F := (|x|^2 + |y|^2)^2 - 2\tilde{F}.$$

F is an isoparametric function such that f has range $[-1, 1]$. Let us set

$$P_0 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P_1 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad u := (x, y)^{tr},$$

where I is the k by k identity matrix. Then F can be rewritten as

$$F = |u|^4 - 2\sum_{i=0}^{1}(\langle P_i u, u\rangle)^2, \quad P_iP_j + P_jP_i = 2\delta_{ij}I.$$

Ferus, Karcher and Münzner's construction is a generalization of this.

Definition 19. *The skew-symmetric (resp., symmetric) Clifford algebra C_n (resp., C'_n) over $\mathbb{R}$ is the algebra generated by the standard basis $e_1, \cdots, e_n$ of $\mathbb{R}^n$ subject to the only constraint*

$$e_ie_j + e_je_i = -2\delta_{ij}I \quad (\textit{resp.}, \ \ e_ie_j + e_je_i = 2\delta_{ij}I).$$

The classification of the Clifford algebras are known:

n	1	2	3	4	5	6	7	8
C_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
δ_n	1	2	4	4	8	8	8	8
C'_n	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
θ_n	2	4	8	8	16	16	16	16

Here, δ_n is the dimension of an irreducible module of C_{n-1}, and θ_n is the dimension of an irreducible module of C'_{n+1}. Moreover, C_n (resp., C'_n) is subject to the periodicity condition $C_{n+8} = C_n \otimes \mathbb{R}(16)$ (resp., $C'_{n+8} = C'_n \otimes \mathbb{R}(16)$). The generators $e_1, \cdots, e_n$ acting on each irreducible module of either C_n or C'_n in the table give rise to n skew-symmetric or symmetric orthogonal matrices $T_1, \cdots, T_n$ satisfying

$$T_iT_j + T_jT_i = \pm 2\delta_{ij}I,$$

a representation of C_n or C'_n on the irreducible module. Note that we have

$$\theta_n = 2\delta_n.$$

This is not fortuitous. It says that we can construct symmetric representations of C'_{m+1} from skew-symmetric representations of C_{m-1}, and vice versa. Indeed, let us be given k irreducible representations $V_1, \cdots, V_k$ of C_{m-1}. Set

$$V := V_1 \oplus + \cdots \oplus V_k \simeq \mathbb{R}^l, \quad l = k\delta_m.$$

The representations of $e_1, \cdots, e_{m-1}$ on $V_1, \cdots, V_k$ give rise to $m-1$ skew-symmetric orthogonal matrices $E_1, \cdots, E_{m-1}$ on V. Set

$$P_0 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P_1 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad P_{1+i} = \begin{pmatrix} 0 & E_i \\ -E_i & 0 \end{pmatrix}, \ 1 \leq i \leq m-1.$$

Then

$$P_i P_j + P_j P_i = 2\delta_{ij} I.$$

$P_0, \cdots, P_m$ give a representation of C'_{m+1} on $\mathbb{R}^{2l}$.

Ferus, Karcher and Münzner's examples, referred to as of OT-FKM type henceforth (OT short for Ozeki and Takeuchi), are

$$F := |u|^4 - 2\sum_{i=0}^{m} (\langle P_i u, u\rangle)^2, \quad u \in \mathbb{R}^{2l}, \quad l = k\delta_m.$$

Note that we recover Nomizu's example when $m = 1$.

By a straightforward calculation, we conclude

Proposition 20. *The two multiplicities $m_\pm$ of an isoparametric hypersurface of OT-FKM type are*

$$(m, k\delta_m - m - 1),$$

where $m, k \in \mathbb{N}$ to make the second entry positive.

We have the following table for the multiplicity pair $(m, k\delta_m - m - 1)$.

$\delta_m =$	1	2	4	4	8	8	8	8	16	$\cdots$
$k = 1$	–	–	–	–	$(5,2)$	$(6,1)$	–	–	$(9,6)$	$\cdots$
$k = 2$	–	$(2,1)$	$(3,4)$	$(4,3)$	$(5,10)$	$(6,9)$	$(7,8)$	$(8,7)$	$(9,22)$	$\cdots$
$k = 3$	$(1,1)$	$(2,3)$	$(3,8)$	$(4,7)$	$(5,18)$	$(6,17)$	$(7,16)$	$(8,15)$	$(9,38)$	$\cdots$
$k = 4$	$(1,2)$	$(2,5)$	$(3,12)$	$(4,11)$	$(5,26)$	$(6,25)$	$(7,24)$	$(8,23)$	$(9,54)$	$\cdots$
$k = 5$	$(1,3)$	$(2,7)$	$(3,16)$	$(4,15)$	$(5,34)$	$(6,33)$	$(7,32)$	$(8,31)$	$(9,70)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Among other things, Ferus, Karcher and Münzner established

Theorem 21. (1) *OT-FKM type with multiplicities on the first, second, fourth columns,* $(4,3)$ *and* $(9,6)$ *are exactly the homogeneous examples, except for the two with multiplicities* $\{2,2\}$ *and* $\{4,5\}$ *not on the list.*

(2) *OT-FKM type with multiplicities on the third and seventh columns are exactly the inhomogeneous examples constructed by Ozeki and Takeuchi.*

So, except for the first, second and fourth columns, we have infinitely many families, each with infinitely many members, of inhomogeneous isoparametric hypersurfaces with four principal curvatures. Note that we also have the fact that OT-FKM type with multiplicities $(2,l)$ is congruent to the one with multiplicities $(l,2)$. Note also that Cartan classified the cases when the multiplicities are $\{1,1\}$ and $\{2,2\}$, both being homogeneous.

Wang [31] investigated the topology of OT-FKM type by K-theory and showed that there are many pairs of minimal isoparametric hypersurfaces in spheres, of identical constant scalar curvature, which are diffeomorphic but noncongruent to each other.

Dorfmeister and Neher [12] in 1985 settled one of the two cases when $g=6$:

Theorem 22. *An isoparametric hypersurface with* $g=6$ *and multiplicities* $m_{\pm}=1$ *is homogeneous.*

5. Development in the 1990s

Another remarkable result, via homotopy theory, was achieved in 1999 by Stolz [29], who classified all the possible multiplicity pairs (m_1,m_2), where $m_1 \le m_2$, of isoparametric hypersurfaces with $g=4$:

Theorem 23. *The multiplicity pairs* $(m_1,m_2), m_1 \le m_2$, *of isoparametric hypersurfaces with four principal curvatures are exactly those in the above table for the OT-FKM type, except for the pairs* $(2,2)$ *and* $(4,5)$ *not in the table.*

He established that if $(m_1,m_2), m_1 \le m_2$, is neither $(2,2)$ nor $(4,5)$, then m_1+m_2+1 is a multiple of $2^{\phi(m_1-1)}$, where $\phi(n)$ denotes the number of natural numbers $s, 1 \le s \le n$, such that $s \equiv 0,1,2,4 \pmod 8$. One can see easily that such pairs (m_1,m_2) are exactly those for the OT-FKM type in the above table.

His approach is reminiscent of the theorem of Adams [1]:

Theorem 24. *If there are* k *independent vector fields on* S^n, *then* $n+1$ *is a multiple of* $2^{\phi(k)}$.

The core technique Adams developed for proving the above theorem on vector fields was to what Stolz reduced his proof.

6. Development in the 2000s

The author and his collaborators [7] and later the author [8], [10] established the following:

Theorem 25. *When $g = 4$, except possibly for the case with multiplicity pair $\{7, 8\}$, an isoparametric hypersurface is, up to congruence, either of OT-FKM type, or homogeneous of multiplicity pair $\{2, 2\}$ or $\{4, 5\}$.*

The proof utilizes commutative algebra, algebraic geometry and Stolz's multiplicity result.

Miyaoka [19], [21] recently settled the other case when $g = 6$:

Theorem 26. *An isoparametric hypersurface with $g = 6$ and multiplicities $m_\pm = 2$ is homogeneous.*

She also gave a simpler and more geometric proof of the theorem of Dorfmeister and Neher by the same technique [18], [20].

So as it stands now, only the case with $g = 4$ and $\{m_+, m_-\} = \{7, 8\}$ remains open, for which we know three inhomogeneous examples of OT-FKM type.

7. Idea of attack on the classification for $g = 4$

We outline the simpler and more powerful method employed in [8] and [10]. In the expansion formula

$$
\begin{aligned}
F(tx + y + w) &= t^4 + (2|y|^2 - 6|w|^2)t^2 + 8(\sum_{a=0}^{m_+} p_a w_a)t \\
&+ |y|^4 - 6|y|^2|w|^2 + |w|^4 - 2\sum_{a=0}^{m_+}(p_a)^2 + 8\sum_{a=0}^{m_+} q_a w_a \\
&+ 2\sum_{a,b=0}^{m_+} \langle \nabla p_a, \nabla p_b \rangle w_a w_b.
\end{aligned}
$$

of Ozeki and Takeuchi, the second components of the 2nd and 3rd fundamental forms are highly convoluted in ten equations. The first three are that the shape operator S_n satisfies $(S_n)^3 = S_n$. Set

$$
< f, g > := \langle \nabla(f), \nabla(g) \rangle, \quad G = \sum_a p_a^2.
$$

Then we have the remaining seven equations:

$$
\begin{aligned}
&< p_a, q_a >= 0;\\
&< p_a, q_b > + < p_b, q_a >= 0, \quad a \neq b;\\
&<< p_a, p_b >, q_c > + << p_c, p_a >, q_b > + << p_b, p_c >, q_a >= 0, \quad a, b, c \text{ distinct};\\
&\sum_a p_a q_a = 0;\\
&16(\sum_a q_a^2) = 16G(\sum_i y_i^2) - < G, G >;\\
&8 < q_a, q_a >= 8(< p_a, p_a > (\sum_i y_i^2) - p_a^2) + << p_a, p_a >, G > -24G - 2\sum_b < p_a, p_b >^2;\\
&8 < q_a, q_b >= 8(< p_a, p_b > (\sum_i y_i^2) - p_a p_b) + << p_a, p_b >, G > -2\sum_c < p_a, p_c >< p_b, p_c > .
\end{aligned}
$$

Here, y_i are the coordinates of y. They seem to be too overwhelming to solve! But,

every cloud has a silver lining.

The 4th of these gory equations is a syzygy equation:

$$\sum_{a=1}^{m_1} p_a q_a = 0.$$

What happens when the homogeneous $p_0, \cdots, p_{m_1}$ of degree 2 form a regular sequence, for which the syzygy is trivial, so that

$$q_a = \sum_b r_{ab} p_b,$$

where $r_{ab} = -r_{ba}$ are homogeneous of degree 1? (That is, we are looking into complete intersections.)

Moreover, what conditions guarantee that $p_0, \cdots, p_{m_1}$ form a regular sequence?

The answer to the second question is Serre's criterion of normal varieties [13]:

Theorem 27. (Special Case) *Over the complex numbers let $p_1, \cdots, p_s$ be a regular sequence of homogeneous polynomials in a polynomial ring, let V be the variety cut out by $p_1, \cdots, p_s$, and let J be the subvariety of V where the rank of the Jacobian of $p_1, \cdots, p_s$ is $< s$. If $\dim(J) \leq \dim(V) - 2$, then the ideal $(p_1, \cdots, p_s)$ is prime.*

Corollary 28. ([7]) *Over the complex numbers, let $p_1, \cdots, p_k, k \geq 2$, be linearly independent homogeneous polynomials of equal degree ≥ 1 in* *a polynomial ring. For $i \leq k$, let V_i be the variety cut out by $p_1, \cdots, p_i$, and let J_i be the subvariety of V_i where the rank of the Jacobian matrix of $p_1, \cdots, p_i$ is $< i$. If $\dim(J_i) \leq \dim(V_i) - 2$ for $1 \leq i \leq k-1$, then $p_1, \cdots, p_k$ form a regular sequence in the polynomial ring.*

The answer to the first question is then facilitated by the following:

Theorem 29. ([8]) *Let $\{m_1, m_2\}$ be the multiplicity pair of an isoparametric hypersurface with four principal values in the sphere. If $m_1 < m_2$ and $p_0, \cdots, p_{m_1}$ form a regular sequence, then the hypersurface is of OT-FKM type.*

Theorem 25 follows from Corollary 28 and Theorem 29.

In fact, when $m_2 \geq 2m_1 - 1$, one can show that indeed $p_0, \cdots, p_{m_1}$ do form a regular sequence by Corollary 28, so that the isoparametric hypersurface is of OT-FKM type, which had been proven in [7] by a considerably more complicated method that does not seem to extend to the remaining cases. By Stolz's multiplicity result, this takes care of exactly the multiplicity pairs except for the anomalous ones $\{3,4\}, \{4,5\}, \{6,9\}, \{7,8\}$, for which $p_0, \cdots, p_{m_1}, m_1 < m_2$, *do not* form a regular sequence in general. The zero locus of a nonregular sequence, even over complex numbers, can be wildly untamed. However, one can now employ the notion of *Condition A* of Ozeki and Takeuchi [25]:

Definition 30. *A focal submanifold of an isoparametric hypersurface with four principal values in the sphere is said to be of Condition A, if its shape operator has a fixed kernel in all normal directions at some point.*

The focal submanifold satisfying Condition A must be of codimension $2, 4$, or 8 in the sphere [25], so that, in particular, the associated isoparametric hypersurface cannot be of multiplicity pair $\{4,5\}$ or $\{6,9\}$. On the other hand, the isoparametric hypersurfaces of OT-FKM type with multiplicity pair $\{3,4\}$ or $\{7,8\}$ do admit points of Condition A on the focal submanifold with codimension 4 or 8 [25], [14]. Conversely, it is proved in [11] that the existence of points of Condition A implies that the isoparametric hypersurface is of OT-FKM type (see also [9]).

With Condition A and Theorem 29, one can now conclude [8], [10] that the isoparametric hypersurface is either homogeneous in the $\{4,5\}$ case, or is of OT-FKM type in the $\{3,4\}$ and $\{6,9\}$ cases, for which the codimension 2 estimates in Corollary 28 is manageable. The essential point is that, for the three multiplicity pairs $\{3,4\}, \{4,5\}$, and $\{6,9\}$, nonexistence of points of Condition A (even locally) implies that either

$p_0, \cdots, p_{m_1}, m_1 < m_2$, form a regular sequence, so that the isoparametric hypersurface is of OT-FKM type, or the 2nd and, hence, the 3rd fundamental forms of the hypersurface coincide with those of the homogeneous example.

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