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RIGIDITY OF SUPERMINIMAL SURFACES IN COMPLEX PROJECTIVE SPACES

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Abstract. This paper solves the rigidity problem for branched superminimal immersions in complex projective spaces. Bertini's theorem and Chow varieties in algebraic geometry as well as results on Teichmüller spaces and the deformation theory of holomorphic maps and used.

1. Introduction. The rigidity phenomenon of superminimal surfaces in CP^n was studied by the first author in [2]. One sees that certain classes of nonsingular plane algebraic curves as well as their quadratic transformations do generate rigid superminimal immersions in CP^2. It was also proved that superminimal immersions generated by a generic rational curve in CP^2 of any degree is rigid. For superminimal immersions generated by plane cubics, rigidity up to finiteness was shown. These supporting evidences lead naturally to the question: Is an arbitrary superminimal surface in CP^n rigid?

Following the line of thought in [2] together with further use of algebraic geometry, we are able to solve positively the above rigidity problem under general conditions. More precisely, we have the following:

THEOREM 1. *Any compact branched superminimal immersion $M \to CP^n$ is rigid up to finiteness. In other words, there are at most finitely many such immersions in CP^n which are isometric to the first one but mutually inequivalent under the isometry of CP^n. Furthermore, the number of possible inequivalent immersions is bounded by a constant depending only on the area of M.*

THEOREM 2. *Let $f: M \to CP^n$ be any holomorphic map, where M is a compact Riemann surface. To a generic A in $PGL(n+1, C)$ regarded as a real algebraic variety, all the branched superminimal immersions generated by Af are rigid.*

Slightly more general versions of the above theorems can be found at the end of Sections 4 and 5. When $n=2$, letting the Riemann surface to vary in the moduli space, we obtain:

THEOREM 3. *Branched Superminimal immersions of genus g, degree d and area $m\pi$*

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in CP^2 form an analytic space with a natural action of $PGL(3, C)$, which contains a real Zariski open dense set consisting of rigid superminimal surfaces. Moreover, this Zariski open dense set intersects each orbit of the $PGL(3, C)$ action in a real Zariski open dense set of the orbit.

Our approach is motivated by an argument of [2], where the generic rational curves in CP^2 was treated. (The proof of Lemma 2 in [2] is incomplete. Although the present paper supersedes it, see the last remark in Section 6.) Let $f, g: M \to CP^2$ be two plane curves which generate branched superminimal immersions ∂f and ∂g, and let F, G be two global meromorphic lifts of f and g into C^3. When ∂f is isometric to ∂g, it was proved in [2] that

$$\|F\|^2 \|F \wedge F'\|^2 = \|G\|^2 \|G \wedge G'\|^2 |r|^2 ,$$

where r is some meromorphic function, and $\| \cdot \|^2$ is the Hermitian norm of C^3. To prove the rigidity for ∂f, it suffices to assert $\|F\|^2 = \|G\|^2$. When M is the Riemann sphere, one can choose F, G to be polynomials in z. If $\|F(z)\|^2 = \langle F(z), F(z) \rangle$ is regarded as a polynomial of two independent variables z and $\bar{z}$, the unique factorization theorem of the polynomial ring $C[z, \bar{z}]$ gives the rigidity if both $\|F(z)\|^2$ and $\|F(z) \wedge F'(z)\|^2$ are irreducible. This was the approach adopted in [2].

In this paper, we will consider M of arbitrary genus g. Choose a global meromorphic lift of f, now denoted $\tilde{f}$. One can interpret $\|\tilde{f}\|^2$ as an algebraic function over $M \times \bar{M}$. Since the ring of algebraic functions lacks unique factorization in general, one studies instead the zero divisor N_f of the function $\|\tilde{f}\|^2$ and applies the unique factorization theorem for divisors. It can be prove that N_f determines f up to unitary equivalence. With this N_f, we obtain: **A**. Unique factorization of $N_f \Rightarrow$ rigidity up to finiteness; **B**. Quantitative properties of $N_f \Rightarrow$ numerical estimate of the "finiteness"; **C**. N_f being prime $\Rightarrow$ rigidity. **A** and **B** immediately give Theorem 1 and Theorem 1′ in Section 4. **C** will lead to Theorem 2, Theorem 2′ of Section 5 and Theorem 3 in Section 6.

To prove Theorem 2, one first proves that N_{Af} is prime for a generic projective transformation $A \in PGL(n+1, C)$ (generic in the sence of real algebraic variety); since "being prime" is an open condition and $PGL(n+1, C)$ is irreducible, it suffices to find a single A_0 such that $N_{A_0 f}$ is prime. Such an A_0 is furnished by Bertini's theorem. One then proves that $N_{(Af)_k}$ are generically prime for all the k-th associated curves $(Af)_k$ of Af. The curve f_{n-1} is taken care of by the duality between CP^n and its dual space $(CP^n)^\wedge$. The case of a general f_k is reduced to the case of f_{n-1} by a suitable projection π: $CP^n \to CP^{k+1}$. Theorem 3 then follows from Theorem 2 and the application of the theory of Teichmüller spaces and the deformation theory of holomorphic maps.

2. Preliminaries.

1. Superminimal surfaces in CP^n. Given a compact Riemann surface M and a nondegenerate holomorphic map $f: M \to CP^n$, let $F(z) = (F_0(z), \ldots, F_n(z))$ be a local

holomorphic lift of f into C^{n+1}. The Gram-Schmidt process applied to the derivatives $F, F', \dots, F^{(k)}$ yields a quantity $\partial^k F$ which is perpendicular to $F, F', \dots, F^{(k-1)}$. The projectivization of $\partial^k F$ defines $\partial^k f$, the branched superminimal immersion of location k generated by the holomorphic curve f. Let f_k be the k-th associated holomorphic curve of f in CP^{n_k}, where

$$n_k = \binom{n+1}{k+1} - 1 . \tag{2.1}$$

It is shown in [3] that the induced metrics satisfy

$$(\partial^k f)^* \langle \, , \, \rangle_{CP^n} = f^*_{k-1} \langle \, , \, \rangle_{CP^{n_{k-1}}} + f^*_k \langle \, , \, \rangle_{CP^{n_k}} . \tag{2.2}$$

In particular (2.2) implies that the holomorphic map

$$f_{k-1} \otimes f_k : M \xrightarrow{f_{k-1} \times f_k} CP^{n_{k-1}} \times CP^{n_k} \xrightarrow{\text{Segre imbedding}} CP^{(n_{k-1}+1)(n_k+1)-1} \tag{2.3}$$

has the same induced metric as $\partial^k f$. For later purpose, let $f^*_k \langle \, , \, \rangle_{CP^{n_k}} = q_k^2 ds^2$, where ds^2 is any fixed metric consistent with the conformal structure of M. Then we have $q_{-1} = q_n = 0$, $q_k \not\equiv 0$ for $0 \leq k \leq n-1$ by the nondegeneracy of f in CP^n, and

$$\Delta \log q_k = K + 2(q_{k-1}^2 + q_{k+1}^2 - 2q_k^2) , \tag{2.4}$$

where Δ and K are the Laplacian and the Gaussian curvature of the metric ds^2.

2. Chow varieties and Bertini's Theorem. Consider the set of irreducible projective varieties $X \subset CP^L$ of fixed degree d and dimension k. It can be proved (cf. [6][11]) that this set, denoted $C_{L,k,d}$, may be given the structure of a quasi-projective variety. Moreover, one can add to this set all the formal sums $X = m_1 X_1 + \cdots + m_l X_l$, where each X_i has degree d_i and $d = m_1 d_1 + \cdots + m_l d_l$. The formal sums X may be viewed as reducible subvarieties of degree d. The enlarged set denoted by $\bar{C}_{L,k,d}$ is a projective variety which contains $C_{L,k,d}$ as a Zariski open set. We call both $C_{L,k,d}$ and $\bar{C}_{L,k,d}$ Chow varieties.

To prove Theorem 3 in Section 6, one needs to prove that a specific set of divisors $N_f \in M \times \bar{M}$ is generically prime, where f varies in a family of holomorphic maps from compact Riemann surfaces M to CP^n. This will be done by an argument similar to the one for Theorem 2 outlined at the end of Section 1. The fact that M itself also varies in the Riemann moduli space M_g can be handled by a result of Teichmüller theory which enables one to realize all the $N_f \in M \times \bar{M}$ as curves in a common projective space. Chow varieties then furnish the conceptual framework for the rest of the arguments.

As we will see, in the proof of Theorem 2 the set of N_f is contained in a fixed linear system, Bertini's Theorem suffices for the arguments there (cf. [6]):

Bertini's Theorem. *Let V be an irreducible subvariety of dimension* ≥ 2 in CP^n, *and let L be the linear system on V defined by the hyperplane cut. Then the prime elements* *form a Zariski open set in L.*

3. The Divisor N_f. Let $f: M \to CP^n$ be a holomorphic map from a compact Riemann surface into CP^n with the Fubini-Study metric, and let $\tilde{f}: M \to C^{n+1}$ be a meromorphic lift of f.

DEFINITION 3.1. To every meromorphic lift $\tilde{f}(p) = (f_0(p), \ldots, f_n(p))$ of f, consider the order of the meromorphic functions f_i at point p, denoted $\mathrm{ord}_p(f_i)$. Define the order

$$\mathrm{ord}_p(\tilde{f}) = \min_{0 \le i \le n} \mathrm{ord}_p(f_i) ,$$

where if $f_i(p) \equiv 0$, we set $\mathrm{ord}_p(f_i) = \infty$. Define the divisor $(\tilde{f}) = \sum_{p \in M} \mathrm{ord}_p(\tilde{f}) p$.

Let $\bar{M}$ denote the surface M with the conjugate complex structure, and let $\bar{p} \in \bar{M}$ be the point corresponding to $p \in M$. We can similarly consider $\overline{CP^n}$ and the map $\bar{f}: \bar{M} \to \overline{CP^n}$ defined by $\bar{f}(\bar{p}) = \overline{f(p)}$, which is holomorphic. Consider a divisor D on the variety $M \times \bar{M}$ and its unique fractorization $D = \sum_{i=1}^{\alpha} m_i D_i$, where $D_1, \ldots, D_\alpha$ are distinct prime divisors.

DEFINITION 3.2. If all the D_i's are of the form $p \times \bar{M}$ or $M \times \bar{q}$ in the above decomposition, D is said to be transversal. If none of the D_i's are of the form $p \times \bar{M}$ or $M \times \bar{q}$, D is said to be skew.

Each divisor D can be decomposed uniquely into $D = D_{\mathrm{sk}} + D_{\mathrm{tr}}$, where the first and second terms are skew and transversal, respectively.

DEFINITION 3.3. Given a meromorphic lift $\tilde{f}$, define a meromorphic function $\|\tilde{f}\|^2$ on $M \times \bar{M}$ by

$$\|\tilde{f}\|^2 = \|\tilde{f}\|^2(p, \bar{q}) = \langle \tilde{f}(p), \tilde{f}(q) \rangle = f_0(p)\overline{f_0(q)} + \cdots + f_n(p)\overline{f_n(q)} .$$

Note that the symbol $\|\cdot\|^2$ does not represent an actual norm square.

LEMMA 3.1. *In the divisor decomposition* $(\|\tilde{f}\|^2) = (\|\tilde{f}\|^2)_{\mathrm{sk}} + (\|\tilde{f}\|^2)_{\mathrm{tr}}$, *we have*

(i) $(\|\tilde{f}\|^2)_{\mathrm{sk}} \ge 0$, *namely, the skew part of the divisor of the function* $\|\tilde{f}\|^2$ *is effective.*

(ii) $(\|\tilde{f}\|^2)_{\mathrm{tr}} = (\tilde{f}) \times \bar{M} + M \times (\bar{\tilde{f}})$.

PROOF. (i) The polar divisor of $\|\tilde{f}\|^2(p, \bar{q})$ can only be of the form $\bigcup_i (p_i \times \bar{M})$ and $\bigcup_i (M \times \bar{q}_i)$, which is transversal.

(ii) Take a local coordinate $z(p)$ around $p_0 \in M$ with $z(p_0) = 0$, we have $\tilde{f}(z) = z^m h(z)$, where h is holomorphic and nonvanishing at p_0, and $m = \mathrm{ord}_{p_0}(\tilde{f})$. Using the notation $\bar{w} = z(\bar{q})$, we have

$$\|\tilde{f}\|^2(p, \bar{q}) = z^m \bar{w}^m \|h\|^2(p, \bar{q})$$

around $(p_0, \bar{p}_0) \in M \times \bar{M}$, where $\|h\|^2(p, \bar{q})$ is holomorphic and positive at $(p_0, \bar{p}_0)$. This implies $\|\tilde{f}\|^2\ (p, \bar{q})$ has order m along $p_0 \times \bar{M}$. q.e.d.

It is easy to see that the divisor $(\|\tilde{f}\|)_{\rm sk}$ is independent of the choice of $\tilde{f}$, which allows:

DEFINITION 3.4. Let $f: M \to \boldsymbol{CP}^n$ be a holomorphic map, and let $\tilde{f}$ be any meromorphic lift of f. Define N_f to be $(\|\tilde{f}\|^2)_{\rm sk}$.

N_f may also be defined invariantly as follows. Consider the map

$$M \times \bar{M} \xrightarrow{f \times \bar{f}} \boldsymbol{CP}^n \times \overline{\boldsymbol{CP}^n} \xrightarrow{\text{Segre}} \boldsymbol{CP}^N, \tag{3.1}$$

where $N=(n+1)^2-1$ and the Segre embedding is given by $x_{ij}=z_i\bar{w}_j$ in homogeneous coordinates. Consider the hyperplane $\boldsymbol{S}_0$ in $\boldsymbol{CP}^N$ defined by the homogeneous equation $x_{00}+\cdots+x_{nn}=0$, which is invariant up to a unitary transformation of $\boldsymbol{CP}^N$ of the form $U \otimes \bar{U}$ naturally induced by a unitary transformation U in $\boldsymbol{CP}^n$. One can verify the following lemma without difficulty:

LEMMA 3.2. *The pullback of $\boldsymbol{S}_0$ as a divisor on $M \times \bar{M}$ through the map in* (3.1) *is* N_f.

We will use the two notions of N_f interchangeably, whichever is convenient. The basic fact about N_f is that it determines f up to unitary equivalence. To prove this, we need:

LEMMA 3.3. *Let M, N be two compact Riemann surfaces, and let F be a meromorphic function on $M \times N$ such that the divisor of F is transversal, namely $(F)_{\rm sk}=0$. Then there exist meromorphic functions u and v defined on M and N respectively such that $F(p, q)=u(p)v(q)$.*

PROOF. Let $(F)=\sum k_i(p_i \times N)+\sum l_j(M \times q_j)$. Choose $q_0 \in N \setminus \bigcup \{q_j\}$. The function $F(p, q_0)$ is holomorphic without zeros whenever p is not any of p_i's. Around (p_i, q_0) one can introduce a local coordinate system $(z(p), w(q))$ with $z(p_i)=0$ and $w(q_0)=0$. Since F has order k_i along $p_i \times N$, we have $F(p, q)=z^{k_i}(p)h(p, q)$, where h is holomorphic and nonvanishing at (p_i, q_0). This implies that the divisor of $F(p, q_0)$ as a function of p is $\sum k_i p_i$.

Let us define $u(p)=F(p, q_0)$ and similarly $v(q)=F(p_0, q)$ for a suitable p_0. Then we have $(u)=\sum k_i p_i$ and $(v)=\sum l_j q_j$. The function $u(p)v(q)$ on $M \times N$ has consequently the same divisor as F, which implies $F(p, q)=Ru(p)v(q)$, where R is a constant. q.e.d.

LEMMA 3.4. *N_f determines f up to unitary transformations. More precisely, if we have two holomorphic maps f and g from M to $\boldsymbol{CP}^n$ such that $N_f=N_g$, then there exists a unitary transformation U from $\boldsymbol{CP}^n$ to itself such that $f=U \circ g$.*

PROOF. Take two meromorphic lifts $\tilde{f}$ and $\tilde{g}$ for f and g. Then by the definition of N_f and the assumption that $N_f=N_g$ we have $(\|\tilde{f}\|^2)_{\rm sk}=(\|\tilde{g}\|^2)_{\rm sk}$, namely,

$(\|\tilde{f}\|^2/\|\tilde{g}\|^2)_{sk}=0$. By Lemma 3.3, there are two functions $u(p)$, $v(q)$ over M such that

$$\|\tilde{f}\|^2/\|\tilde{g}\|^2=u(p)\overline{v(q)}\,.$$

Let $p=q$. The left hand side becomes a true norm square and therefore real. Hence $u(p)\overline{v(p)}$ is a real function, namely

$$u(p)\overline{v(p)}=\overline{u(p)}v(p)\,, \qquad u(p)/v(p)=\overline{u(p)/v(p)}\,.$$

The meromorphic function $u(p)/v(p)$ is real, and therefore is a non-zero real constant R. This implies

$$\|\widetilde{f(p)}\|^2=R\|\widetilde{g(p)}\|^2|v(p)|^2\,,$$

where the symbol $\|\cdot\|^2$ is understood to be an actual norm square. So we have

$$dd^c\log\|\widetilde{f(p)}\|^2=dd^c\log\|\widetilde{g(p)}\|^2\,,$$

which means that f and g are isometric. By Calabi's rigidity result for holomorphic curves, f and g are unitarily equivalent. q.e.d.

Now let f_1, f_2 be two holomorphic maps from M to CP^m and CP^n. We can define another holomorphic map $f_1\otimes f_2\colon M\to CP^{(m+1)(n+1)-1}$ by projectivizing

$$\tilde{f}_1\otimes\tilde{f}_2(p)=(\dots z_i(p)\omega_j(p)\dots)\,,$$

where $\tilde{f}_1(p)=(z_0(p),\dots,z_m(p))$ and $\tilde{f}_2(p)=(w_0(p),\dots,w_n(p))$ are the meromorphic lifts of f_1 and f_2, respectively. Taking the skew part of the divisor of the identity $\|\tilde{f}_1\otimes\tilde{f}_2\|^2=\|\tilde{f}_1\|^2\|\tilde{f}_2\|^2$, we get:

LEMMA 3.5. $N_{f_1\otimes f_2}=N_{f_1}+N_{f_2}$.

What we have done so far is sufficient to prove a general rigidity theorem up to finiteness. However, to give explicit upper bounds, we need to determine the homology class represented by N_f in $H_2(M\times\bar{M})$. By the Künneth formula, we have direct sum decomposition

$$H_2(M\times\bar{M})=H_2(M)\otimes H_0(\bar{M})+H_0(M)\otimes H_2(\bar{M})+H_1(M)\otimes H_1(\bar{M})\,.$$

The first two direct components are isomorphic to $\boldsymbol{Z}$ and their generators are represented by cycles of the form $M\times\bar{q}$ and $p\times\bar{M}$ respectively, which we denote by $[M]$ and $[\bar{M}]$. The class represented by N_f is denoted by $[N_f]$.

LEMMA 3.6. $[N_f]=\deg(f)([M]+[\bar{M}])$.

PROOF. We will use the invariant definition of N_f given in Lemma 3.2. Consider the sequence of maps in (3.1). We may assume without loss of generality that $f(M)$ is not contained in the hyperplane $H_0\subset CP^n$ defined by $z_0=0$. The divisor $\boldsymbol{S}_0$ in CP^N is linearly equivalent to the hyperplane defined by $x_{00}=0$, which is pulled back through the above sequence of maps to $f^*(H_0)\times\bar{M}+M\times\bar{f}^*(\bar{H}_0)$. N_f, being the pullback of

S_0, is then linearly equivalent to (and hence homologous to) $f^*(H_0)\times \bar{M}+M\times \bar{f}^*(\bar{H}_0)$. Our lemma then follows from the fact that $\deg(f^*(H_0))=\deg(f)$. q.e.d.

The next three lemmas are concerned with the decomposition of N_f.

LEMMA 3.7. *Let $N_f=\sum_{i=1}^{\alpha} m_i N_i$ be the decomposition of N_f into its prime components. Then to each component N_i, there are integers $a_i\geq 1$ and $b_i\geq 1$ such that*

$$[N_i]=a_i[M]+b_i[\bar{M}]+Q_i\,,$$

in $H_2(M\times \bar{M})$, where $Q_i\in H_1(M)\otimes H_1(\bar{M})$.

PROOF. From the decomposition

$$H_2(M\times \bar{M})=H_2(M)\otimes H_0(\bar{M})+H_0(M)\otimes H_2(\bar{M})+H_1(M)\otimes H_1(\bar{M})\,,$$

we have $[N_i]=a_i[M]+b_i[\bar{M}]+Q_i$. By definition, the divisor N_f is skew, therefore, all the N_i are skew. We can therefore apply the natural projection $M\times\bar{M}\to M$, which sends N_i to M. Because N_i is an analytic subvariety, the above projection from N_i to M has a positive degree, which is just a_i. So both a_i and b_i are positive. q.e.d.

LEMMA 3.8. *Under the above notation, let n_0 be the number of different decompositions $N_f=D_1+D_2$, where both D_1 and D_2 are non-negative divisors. Then we have $n_0\leq 2^{\deg(f)}$. Here the interchange of D_1 and D_2 are counted as different decompositions.*

PROOF. Consider the factorization $N_f=\sum_{i=1}^{\alpha} m_i N_i$ where all N_i are distinct prime divisors and $m_i\geq 1$. By Lemmas 3.6 and 3.7, we have $[N_f]=\deg(f)([M]+[\bar{M}])$ and $[N_i]=a_i[M]+b_i[\bar{M}]+Q_i$ with $a_i\geq 1$ and $b_i\geq 1$; therefore $\deg(f)=m_1a_1+\cdots+m_\alpha a_\alpha$. This implies $\alpha\leq\deg(f)$. In the worst case, all the m_i and a_i are 1, and there are exactly $2^{\deg(f)}$ different decompositions. q.e.d.

For application to the rigidity problems, one only needs the decomposition of the form $N_f=N_{f_1}+N_{f_2}$. This allows a refinement of the previous lemma. Observe that $M\times\bar{M}$ has an anti-holomorphic automorphism τ defined by τ: $(p,\bar{q})\to(q,\bar{p})$, under which N_f is invariant. Similarly, N_{f_1} and N_{f_2} are also invariant under τ. Hence, we can regroup the decomposition $N_f=\sum_{i=1}^{\alpha} m_i N_i$ as follows. If N_i is invariant under τ, we keep the term m_iN_i; otherwise, we group it with its image under τ. To avoid too much notation, we will still use $N_f=\sum_{i=1}^{\alpha} m_i N_i$ to denote the new decomposition. As a result of this regrouping, $[N_i]=a_i([M]+[\bar{M}])+Q_i$, where a_i is a positive integer and Q_i is as in Lemma 3.7.

LEMMA 3.9 *Assume that the Riemann surface M does not have nontrivial automorphisms. Let n_0 be the number of different decompositions $N_f=N_{f_1}+N_{f_2}$. Then $n_0\leq 2^{\deg(f)/2+1}$.*

PROOF. From the argument above, we have $[N_i]=a_i([M]+[\bar{M}])+Q_i$. If there is an i such that $a_i=1$, then N_i is a (1, 1) correspondence between Riemann surfaces M and $\bar{M}$ (for more about correspondence, see [5]). Therefore, $N_i \subset M \times \bar{M}$ defines an isomorphism T_i: $M \to \bar{M}$. If there is another index j such that $a_j=1$, we will similarly have a T_j, and so $T_i^{-1} \circ T_j$ is an automorphism of M. By the assumption of the lemma, we have $T_i=T_j$ and therefore $N_i=N_j$. So we have proved that all a_i's are greater than 1 with at most one exception. In the worst case, let $a_1=1$ and $a_2=\cdots=a_\alpha=2$ and $m_2=\cdots=m_\alpha=1$. Then we have $N_f=m_1N_1+N_2+\cdots+N_\alpha$, and $\deg(f)=m_1+2(\alpha-1)$. The number of decompositions n_0 then satisfies

$$n_0 \le (m_1+1)2^{\alpha-1} \le (m_1+1)2^{(\deg(f)-m_1)/2} \le \frac{m_1+1}{2^{m_1/2}} 2^{\deg(f)/2} \le 2^{\deg(f)/2+1} .$$

q.e.d.

REMARK. In Lemmas 3.8 and 3.9, if we count the exchange of D_1 and D_2 as well as N_{f_1} and N_{f_2} as the same decomposition, the number of different decompositions n_0 can be reduced in half.

4. Rigidity up to Finiteness. We will prove Theorem 1 in this section.

LEMMA 4.1. *Let $\partial^k f$ and $\partial^l g$ be two branched superminimal immersions form M to CP^n. Then we have*

(i) $N_{f_{k-1}}=N_{g_{l-1}}$ *and* $N_{f_k}=N_{g_l} \Leftrightarrow \partial^k f$ *and* $\partial^l g$ *differ only by a unitary transformation of* CP^n;

(ii) $N_{f_{k-1}}=N_{g_l}$ *and* $N_{f_k}=N_{g_{l-1}} \Leftrightarrow \partial^k f$ *and* $\partial^l g$ *differ only by an orientation reversing isometry of* CP^n.

PROOF. The "$\Leftarrow$" part is obvious. We will prove "$\Rightarrow$". Let us begin with (i). If $N_{f_{k-1}}=N_{g_{l-1}}$ and $N_{f_k}=N_{g_l}$, then by Lemma 3.4, f_{k-1} and g_{k-1} are unitarily equivalent and therefore isometric to each other. The same is true for f_k and g_k. So by the recursion formula (2.4), we easily see that f_{k-i} and g_{l-i} are isometric to each other for all i. Now, by the fact that $q_{-1}=0$ but $q_k \neq 0$ for $0 \le k \le n-1$, we have $k=l$, and f and g isometric. By Calabi's the rigidity theorem for holomorphic curves, we are done.

To prove (ii), we use the recursion formula inductively in reverse order and conclude that f_{k-1-i} and g_{l+i} are isometric for all i. When i goes from 0 upward, f_{-1} is the first f_{k-1-i} that vanishes, which implies that $g_{k+l}=0$ is the first g_{l+i} to vanish. This means that g lies in a $CP^{k+l} \subset CP^n$ as a non-degenerate curve. In the above induction, letting $i=k-1$, we see that $f=f_0$ is isometric to g_{k+l-1} and therefore unitarily equivalent to it. Notice that g_{k+l-1} may be regarded as a curve in the dual space of the above CP^{k+l}, so that g_{k+l-1} can be canonically realized in CP^{k+l} as an anti-holomorphic curve. Therefore there is an orientation reversing isometry of CP^n which transforms f to g_{k+l-1}. This proves (ii). q.e.d.

REMARK. When (i) or (ii) of the above lemma is true, we will call $\partial^k f$ and $\partial^l g$

unitarily equivalent or isometrically equivalent in CP^n respectively.

Now applying Lemma 3.8 to the curve $f_{k-1} \otimes f_k$, we have $n_0 \leq 2^{\deg(f_{k-1} \otimes f_k)}$. Note that from (2.2), it is easy to see that $\deg(f_{k-1} \otimes f_k) = A(M)/\pi$, where $A(M)$ represents the area of $\partial^k f$ as a branched minimal surface.

THEOREM 1. *Let $\partial^k f$ be a superminimal surface from M to CP^n. Among all the superminimal surfaces isometric to $\partial^k f$, there are at most n_0 unitarily (or $n_0/2$ isometrically) inequivalent ones, where $n_0 \leq 2^{A(M)/\pi}$.*

PROOF. Let $\partial^k f$ and $\partial^l g$ be isometric. By (2.3), $f_{k-1} \otimes f_k$ and $g_{l-1} \otimes g_l$ are isometric and hence unitarily equivalent. Therefore by this unitary equivalence and Lemma 3.5, we have

$$N_{f_{k-1} \otimes f_k} = N_{g_{l-1} \otimes g_l}, \quad N_{f_{k-1}} + N_{f_k} = N_{g_{l-1}} + N_{g_l}.$$

Lemma 3.8 then implies that the divisor on both sides of the previous equation can be decomposed into the sum of two divisors in at most n_0 ways. If there are more than n_0 superminimal surfaces in CP^n which are isometric to each other, at least two of them, still called $\partial^k f$ and $\partial^l g$, will share the same divisor decomposition. In other words $N_{f_{k-1}} = N_{g_{l-1}}$ and $N_{f_k} = N_{g_l}$. By Lemma 4.1, $\partial^k f$ and $\partial^l g$ are unitarily equivalent.

The statement for the isometric equivalence follows from the remark at the end of Section 3. q.e.d.

Similarly, Lemma 3.9 gives:

THEOREM 1′. *Using the same notation as in Theorem* 1 *and further assuming that M is a Riemann surface without automorphisms, we have the sharper estimate $n_0 \leq 2^{A(M)/2\pi + 1}$.*

5. Generic Rigidity. We now prove Theorem 2. By Lemma 4.1 of the previous section, the branched superminimal immersion $\partial^k f : M \to CP^n$ is rigid if the divisors N_{f_k} and $N_{f_{k-1}}$ are prime. On the other hand, Lamma 3.5 says $N_{f_1 \otimes f_2} = N_{f_1} + N_{f_2}$ and we cannot expect to prove that N_f is prime for all f in general. We will however show that for a generic projective transformation $A : CP^n \to CP^n$, $N_{(Af)_k}$ are prime for all the associated curves $(Af)_k$.

We will first handle the cases when $k = 0$ and $n-1$. The other cases will be reduced to the case $k = n-1$ by an appropriate projection $\pi : CP^n \to CP^{k+1}$; for this purpose, we will enlarge the set of projective transformations of CP^n to include linear rational maps. Let $f : M \to CP^n$ be holomorphic, which is not necessarily nondegenerate.

DEFINITION 5.1. Let A: $[z_0, \dots, z_n] \mapsto [\dots \sum_{j=0}^n a_{ij} z_j \dots]$ be a rational map from CP^n to itself, where (a_{ij}) is a non-zero $(n+1) \times (n+1)$ matrix, and $\ker(A) = \{[z_0, \dots, z_n] \in CP^n \mid \sum_{j=0}^n a_{ij} z_j = 0 \text{ for } i = 0, \dots, n\}$. Define $S_n = \{A \mid f(M) \not\subset \ker(A)\}$.

DEFINITION 5.2. Let $A \in S_n$ and $\tilde{f} = (f_0, \dots, f_n)$ be a meromorphic lift of f. Define

Af to be the holomorphic curve with the lift

$$\widetilde{Af}=\left(\sum_{j=0}^{n} a_{0j}f_j, \ldots, \sum_{j=0}^{n} a_{nj}f_j\right).$$

Note that since $\ker(A)\subset CP^n$ is the set where A fails to be regular, $Af\colon M\to CP^n$ is well-defined for $A\in S_n$. We will see that if the above-mentioned π is so chosen that $\ker(\pi)\cap f(M)=\varnothing$ (which implies $\pi\in S_n$), πf can be deformed nicely into an Af with $A\in PGL\,(n+1, C)\subset S_n$. This will be a key point in our reduction argument through π.

LEMMA 5.1. *Using the previous notation, the following are equivalent*: (i) $\ker(A)\cap f(M)=\varnothing$; (ii) $\deg(Af)=\deg(f)$; (iii) $[N_{Af}]=\deg(f)([M]+[\bar{M}])$.

PROOF. (i)$\Leftrightarrow$(ii) is standard in algebraic geometry ([10]). (ii)$\Leftrightarrow$(iii) follows from Lemma 3.6. q.e.d.

Our main objective is to prove that $N_{Af}=(\|\widetilde{Af}\|^2)_{\text{sk}}$ is prime for a generic A. Because

$$\|\widetilde{Af}\|^2(p,\bar{q})=\sum_{i,j,k=0}^{n} a_{ij}\bar{a}_{ik}f_j(p)\overline{f_k(q)}\,, \tag{5.1}$$

we are led to looking at the linear system of functions

$$L_f=\{F_\lambda(p,q)=\sum_{j,k=0}^{n}\lambda_{jk}f_j(p)\overline{f_k(q)}\mid \lambda=(\lambda_{jk})\in CP^{(n+1)^2-1},\ F_\lambda\not\equiv 0\}\,, \tag{5.2}$$

and consider whether $(F_\lambda)_{\text{sk}}$ is prime. Let us first describe L_f in terms of divisors according to a standard construction in algebraic geometry (cf. [11]). It is easy to see that the meromorphic functions $f_j(p)\overline{f_k(q)}$ over $M\times\bar{M}$ define the holomorphic map (see (3.1)).

$$M\times\bar{M}\xrightarrow{f\times\bar{f}} CP^n\times\overline{CP^n}\xrightarrow{\text{Segre}} CP^N\,. \tag{5.3}$$

DEFINITION 5.3. Let H_λ be the hyperplane in CP^N defined by $\sum_{j,k=0}^{n}\lambda_{jk}x_{jk}=0$, and when $\text{Segre}\circ(f\times\bar{f})(M\times\bar{M})\not\subset H_\lambda$, let D_λ be the pullback of H_λ via the map in (5.3) to $M\times\bar{M}$. Define the linear system $\hat{L}_f$ of divisors

$$\hat{L}_f=\{D_\lambda\mid\lambda\in CP^{(n+1)^2-1},\ \text{Segre}\circ(f\times\bar{f})(M\times\bar{M})\not\subset H_\lambda\}\,.$$

Furthermore (cf. [11]), there exists a divisor $\boldsymbol{D}\subset M\times\bar{M}$ such that $D_\lambda=(F_\lambda)-\boldsymbol{D}$ for $D_\lambda\in\hat{L}_f$, $F_\lambda\in L_f$, where $\boldsymbol{D}=\text{g.c.d.}_{jk}(f_j\bar{f}_k)$. In our case, since the divisors $(f_i\bar{f}_j)$ over $M\times\bar{M}$ are transversal, $\boldsymbol{D}$ is also transversal. Therefore we can decompose D_λ into skew and transversal parts as $D_\lambda=(F_\lambda)_{\text{sk}}+((F_\lambda)_{\text{tr}}-\boldsymbol{D})$, where both parts are effective because D_λ is effective.

LEMMA 5.2. $[D_\lambda]=\deg(f)([M]+[\bar{M}])\,.$

PROOF. This follows from an argument similar to that in the proof of Lemma 3.6. q.e.d.

DEFINITION 5.4. Define $P_f \subset \hat{L}_f$ to be the set of all the prime divisors in $\hat{L}_f$.

LEMMA 5.3. *P_f has the following properties:*
(i) *P_f is a dense Zariski open set in $\hat{L}_f$;*
(ii) *$D_\lambda \in P_f \Leftrightarrow (F_\lambda)_{sk}$ is prime and $[(F_\lambda)_{sk}] = \deg(f)([M] + [\bar{M}])$.*

PROOF. (i) Since $\hat{L}_f$ is a hyperplane cut of $M \times \bar{M}$ via the map in (5.3), by Bertini's theorem P_f is a dense Zariski open set.

(ii) "$\Rightarrow$": Given a $D_\lambda \in P_f$, since D_λ is prime, it is either skew or transversal. From $[D_\lambda] = \deg(f)([M] + [\bar{M}])$, we know D_λ can only be skew. The result follows from the decomposition $D_\lambda = (F_\lambda)_{sk} + ((F_\lambda)_{tr} - \boldsymbol{D})$.

(ii) "$\Leftarrow$": From the decomposition $D_\lambda = (F_\lambda)_{sk} + ((F_\lambda)_{tr} - \boldsymbol{D})$, we see $D_\lambda \geq (F_\lambda)_{sk}$. So if $[(F_\lambda)_{sk}] = \deg(f)([M] + [\bar{M}])$, which means if $[D_\lambda] = [(F_\lambda)_{sk}]$, we must conclude that $D_\lambda = (F_\lambda)_{sk}$ because nontrivial effective divisors represent nontrivial homology classes. Hence D_λ is prime by the assumption that $(F_\lambda)_{sk}$ is prime. q.e.d

Lemma 5.3 tells us that $(F_\lambda)_{sk}$ is prime for a generic λ. As we recall from (5.1) and (5.2), when $\lambda = AA^*$, we have $\|\widetilde{Af}\|^2 = F_\lambda$ and thus $N_{Af} = (F_\lambda)_{sk}$. This motivates the following:

DEFINITION 5.5. Define the sequence of maps $S_n \xrightarrow{d} S_n \times \bar{S}_n \xrightarrow{\lambda} \hat{L}_f$ by $d(A) = (A, \bar{A})$ and $\lambda(A, \bar{B}) = D_{AB^*}$. The second map may be seen more explicitly from the formula $\lambda_{jk} = \sum_{i=0}^{n} a_{ij} \bar{b}_{ik}$.

At the pairs where $AB^* = 0$, the map λ is not defined; however this is clearly a Zariski closed set which does not intersect $d(S_n)$ (because AA^* cannot be zero). Therefore we will exclude them from our consideration. Our question now is to find an A such that $\lambda \circ d(A) \in P_f$. In view of Lemma 5.3 (ii), we make:

DEFINITION 5.6. Define $S_n^0 = \{A \in S_n \mid N_{Af}$ is prime, $\deg(Af) = \deg(f)\}$.

From the sequence of the maps defined in Definition 5.5, we have:

LEMMA 5.4. *$S_n^0 = \lambda^{-1} d^{-1}(P_f)$. Equivalently, S_n^0 consists of all the A in S_n such that $[N_{Af}]$ is prime and $[N_{Af}] = \deg(f)([M] + [\bar{M}])$.*

PROOF. This follows by the definitions of S_n^0 and P_f together with Lemma 5.1 and Lemma 5.3 (ii). q.e.d.

LEMMA 5.5. *Consider S_n as a real algebraic variety with its Zariski topology. Then S_n^0 is a dense open set. In particular, in the real algebraic variety $PGL\ (n+1) \subset S_n$, $S_n^0 \cap PGL(n+1)$ is dense open.*

PROOF. In Definition 5.5, if we regard S_n as a real algebraic variety and $S_n \times \bar{S}_n$

and L_f as complex algebraic varieties, then under their Zariski topologies, both d and λ are continuous maps. From Lemma 5.3, $P_f \subset \hat{L}_f$ is open, and therefore $S_n^0 = d^{-1}\lambda^{-1}(P_f)$ is open in S_n. By the irreducibility of S_n, if $d^{-1}\lambda^{-1}(P_f)$ is non-empty, it will also be dense in S_n. Hence we only need to prove $d^{-1}\lambda^{-1}(P_f) \neq \emptyset$, namely $d(S_n) \cap \lambda^{-1}(P_f) \neq \emptyset$. Since $P_f \subset \hat{L}_f \subset CP^{(n+1)^2-1}$ is an open dense set, there exists a $\lambda \in P_f$ such that $\det(\lambda_{jk}) \neq 0$. By defintion $\lambda(A, \bar{B}) = AB^*$, and we choose $(A, \bar{B}) = (\lambda, I) \in S_n \times \bar{S}_n$ to infer that $\lambda^{-1}(P_f) \neq \emptyset$. The continuity of λ implies that $\lambda^{-1}(P_f)$ is also open. Therefore, if $d(S_n) \cap \lambda^{-1}(P_f) = \emptyset$, $d(S_n)$ will be contained in a proper subvariety of $S_n \times \bar{S}_n$. Lifting through homogeneous coordinates, we know that the subset $\{((\ldots, a_{ij}, \ldots), (\ldots, \bar{a}_{ij}, \ldots)) \mid 0 \leq i, j \leq n\} \subset C^{2(n+1)^2}$ is contained in a proper subvariety of $C^{2(n+1)^2}$. This is equivalent to saying that the real linear subspace $R^{2(n+1)^2} \subset C^{2(n+1)^2}$ is contained in a proper subvariety of $C^{2(n+1)^2}$, which is impossible. So we have proved the first statement.

Notice that $PGL(n+1, C) \subset S_n$ is the set of those $A \in S_n$ represented by nonsingular matrices, it is therefore open in S_n. Since $d^{-1}\lambda^{-1}(P_f)$ is dense open in S_n, $PGL(n+1, C) \cap d^{-1}\lambda^{-1}(P_f)$ is non-empty and open in $PGL(n+1, C)$. By the irreducibility of $PGL(n+1, C)$, we have proved the second statement.

Let us now prove a similar statement for the $(n-1)$-th associated curve f_{n-1}.

LEMMA 5.6. *Let $f: M \to CP^n$ be a nondegenerate holomorphic curve, and f_{n-1} its $(n-1)$-th associated curve. Then, the set $\hat{S}_n^0 = \{A \in PGL(n+1, C) \mid N_{(Af)_{n-1}}$ is prime$\}$ is open dense in the real algebraic variety S_n.*

PROOF. Let $(CP^n)^\wedge$ denote the dual space of CP^n. There is an isomorphism between the group of projective transformations of CP^n and that of $(CP^n)^\wedge$, denoted $A \leftrightarrow \hat{A}$. The associated curve $f_{n-1}: M \to (CP^n)^\wedge$ satisfies $(Af)_{n-1} = \hat{A}f_{n-1}$. Applying Lemma 5.5 to the curve f_{n-1}, we are done. q.e.d.

To deal with the f_k's for $k \neq 0, n-1$, we first introduce some notations. Given $A: C^{n+1} \to C^{n+1}$, there is a naturally induced linear transformation $A_k: \bigwedge^k C^{n+1} \to \bigwedge^k C^{n+1}$. Let S_n be given as in Definition 5.1 and let S_{n_k} denote the similar space associated with the curve $f_k: M \to CP^{n_k}$ with n_k given in (2.1). Let $S_{n_k}^+$ be the variety of all linear rational maps from CP^{n_k} to itself. Clearly $S_{n_k} \subset S_{n_k}^+$. We also have a regular map $W_k: S_n \to S_{n_k}^+$ defined by $A \to A_k$.

We easily have:

LEMMA 5.7. *Let $f: M \to CP^n$ be a holomorphic curve with $f_k: M \to CP^{n_k}$ its k-th associated curve. If $A \in S_n$ and $A_k \in S_{n_k}$, then $(Af)_k$ and $A_k f_k$ are well-defined and $(Af)_k = A_k f_k$.*

We also need:

LEMMA 5.8. *Let $f: M \to CP^n$ be a nondegenerate holomorphic map. For each integer $k = 1, \cdots, n$, there exists a projection $\pi: CP^n \to CP^k$ such that* $\deg(\pi f)_{k-1} = \deg(f_{k-1})$.

π may be viewed as an element in S_n.

PROOF. Consider a projection $\pi: CP^n \to CP^k \subset CP^n$. By Lemma 5.7, if π is properly chosen, the corresponding $\pi_{k-1}: CP^{n_{k-1}} \to CP^{n_{k-1}}$ satisfies $(\pi f)_{k-1} = \pi_{k-1} f_{k-1}$. Therefore, it suffices to find a $\pi \in S_n$ such that π_{k-1} preserves the degree of f_{k-1}. By Lemma 5.1, this is equivalent to $f(M) \not\subset \ker(\pi)$ and $\ker(\pi_{k-1}) \cap f_{k-1}(M) = \emptyset$, namely π_{k-1} is regular at every point of $f_{k-1}(M)$. The existence of such a π can be proved easily by the following geometric consideration.

The $(k-1)$-th assocated curve f_{k-1} is just the map which sends each $p \in M$ to the $(k-1)$-th osculating plane of the curve f at p. Therefore we can regard $f_{k-1}(p)$ as a $(k-1)$-plane in CP^n. The projection π_{k-1} induced from the projection π sends a $(k-1)$-plane P uniquely and regularly to another $(k-1)$-plane $\pi_{k-1}(P)$ as long as $P \cap \ker(\pi) = \emptyset$. Therefore, in order to insure that π_{k-1} be regular at all points in $f_{k-1}(M)$, it suffices to have $f_{k-1}(p) \cap \ker(\pi) = \emptyset$ for all $p \in M$. (Note that this also insures that $f(M) \cap \ker(\pi) = \emptyset$ and therefore $\pi \in S_n$.) In other words, we must find a π such that $\ker(\pi)$ does not intersect any $f_{k-1}(p)$. Since $f_{k-1}(p)$ is just a one-parameter family of $(k-1)$-planes depending holomorphically on p, its envelop, namely the set $E = \bigcup_{p \in M} f_{k-1}(p)$ is a k-dimensional algebraic variety of CP^n. Therefore, a generic $(n-k-1)$-plane F does not intersect E. Now choose a π with $\ker(\pi) = F$. q.e.d.

LEMMA 5.9. *Let $f: M \to CP^n$ be a nondegenerate holomorphic curve, and f_{k-1} its $(k-1)$-th associated curve. Then, the set $\{A \in S_n \mid \deg(Af)_{k-1} = \deg(f_{k-1}),\ N_{(Af)_{k-1}}$ is prime$\}$ is open dense in the real algebraic variety S_n. In particular, the set $\{A \in PGL(n+1, C) \mid \deg(Af)_{k-1} = \deg(f_{k-1}),\ N_{(Af)_{k-1}}$ is prime$\}$ is open dense in the real algebraic variety $PGL(n+1, C)$.*

PROOF. The cases of $k=1$ and n are proved in Lemmas 5.5 and 5.6. Let us assume that k takes any other value. We can apply Lemma 5.5 to $f_{k-1}: M \to CP^{n_{k-1}}$. We then have the corresponding dense open subset $S^0_{n_{k-1}} \subset S_{n_{k-1}}$. Consider the regular map $W_{k-1}: S_n \to S^+_{n_{k-1}}$ defined before Lemma 5.7. What we must prove now is equivalent to $W^{-1}_{k-1}(S^0_{n_{k-1}})$ being dense open in S_n. By the irreducibility of S_n, it suffices to prove that $W^{-1}_{k-1}(S^0_{n_{k-1}})$ is nonempty. Equivalently, we need only to find a single $A \in S_n$ such that $N_{A_{k-1} f_{k-1}}$ is prime and $\deg(A_{k-1} f_{k-1}) = \deg(f_{k-1})$. To this end, we know by Lemma 5.8 that there is a projection $\pi \in S_n$ such that $\pi: CP^n \to CP^k \subset CP^n$ and $\deg(\pi_{k-1} f_{k-1}) = \deg(f_{k-1})$. Applying Lemma 5.6 to the curve $(\pi f)_{k-1} = \pi_{k-1} f_{k-1}: M \to (CP^k)^\wedge$, we find a projective transformation B of CP^k such that $N_{(B\pi f)_{k-1}}$ is prime. Since B does not change the degree of a curve in CP^k, $A = B\pi$ will be what we are looking for. q.e.d

THEOREM 2. *Let $f: M \to CP^n$ be any holomorphic map (not necessarily non-degenerate), where M is a compact Riemann surface. To a generic projective transformation A of CP^n, all the superminimal surfaces generated by the holomorphic map Af are rigid.*

PROOF. Assume first that f is nondegenerate. By Lemmas 5.5, 5.6 and 5.9, to each associated curve f_k, $k=0, \dots, n-1$, we have found a set of A in the real algebraic variety $PGL(n+1, \boldsymbol{C})$ such that $N_{(Af)_k}$ is prime. The intersection of these sets, denoted S^0, is still open and dense in $PGL(n+1, \boldsymbol{C})$. For $A \in S^0$, all the $N_{(Af)_k}$ are prime. Take an A in S^0 and consider superminimal surfaces $\partial^k Af$ and $\partial^l g$ generated by Af and g. If $\partial^k Af$ and $\partial^l g$ are isometric to each other, then by (2.3) $(Af)_{k-1} \otimes (Af)_k$ and $g_{l-1} \otimes g_l$ are isometric. As in the proof of Theorem 1, we have

$$N_{(Af)_{k-1} \otimes (Af)_k} = N_{g_{l-1} \otimes g_l}, \quad N_{(Af)_{k-1}} + N_{(Af)_k} = N_{g_{l-1}} + N_{g_l}.$$

Since $A \in S^0$ implies both $N_{(Af)_{k-1}}$ and $N_{(Af)_k}$ are prime, we have either $N_{(Af)_{k-1}} = N_{g_{l-1}}$ and $N_{(Af)_k} = N_{g_l}$, or $N_{(Af)_{k-1}} = N_{g_l}$ and $N_{(Af)_k} = N_{g_{l-1}}$. By Lemma 3.5, $\partial^k Af$ and $\partial^l g$ differ by a unitary transformation or an orientation reversing isometry in $\boldsymbol{C}P^n$.

If f is degenerate, we consider f as a nondegenerate curve in a smaller space $\boldsymbol{C}P^m \subset \boldsymbol{C}P^n$ so that we can find a projective transformation B of $\boldsymbol{C}P^m$ such that $N_{(Bf)_k}$ are prime for $0 \le k \le m-1$. However, extending B to a projective transformation A of $\boldsymbol{C}P^n$, we see that $N_{(Af)_k}$ are just $N_{(Bf)_k}$, which are prime. The existence of this A implies the existence of generic $A \in PGL(n+1, \boldsymbol{C})$ such that $N_{(Af)_k}$ are prime for $0 \le k \le m-1$ by the same principle as we used in the proof of Lemmas 5.5, 5.6 and 5.9. The rest of the argument goes exactly as in the nondegenerate case. q.e.d.

REMARK. If we consider the problem of rigidity in a larger context by allowing different superminimal immersions to have target spaces of different dimensions, there is a slightly more general version of Theorem 2.

THEOREM 2′. Let $f: M \to \boldsymbol{C}P^n$ be any holomorphic map, where M is a compact Riemann surface. To a generic projective transformation A of $\boldsymbol{C}P^n$, all the superminimal surfaces $\partial^k(Af)$ generated by the holomorphic map Af fall into one of the following two categories:

(i) $\partial^k(Af)$ is a holomorphic or anti-holomorphic map and is rigid;

(ii) $\partial^k(Af)$ is neither holomorphic nor anti-holomorphic, and there are exactly two isometrically inequivalent superminimal surfaces that are isometric to it. These two surface are just $\partial^k(Af)$ itself and the holomorphic curve $(Af)_{k-1} \otimes (Af)_k$.

6. Generic Rigidity in Moduli Space. We will construct moduli spaces of superminimal immersions in $\boldsymbol{C}P^2$ on which the generic rigidity is proved. To motivate the construction, notice that the area of a branched superminimal immersion, being an integral mutiple of π, should remain constant in any continuous variation. Secondly, a continuous family of maps $h: M \to \boldsymbol{C}P^2$ are in the same homotopy class, which is equivalent to having the same mapping degree d defined by $h_*([M]) = dH$ with $[M]$ and H the generators of $H_2(M, \boldsymbol{Z})$ and $H_2(\boldsymbol{C}P^n, \boldsymbol{Z})$, respectively (cf. [7]). Furthermore we will allow the compact Riemann surfaces M to vary in the Riemann moduli space M_g, where g is the genus.

In CP^2, besides the trivial case of holomorphic or anti-holomorphic curves, superminimal immersions ∂f are in one to one correspondence with pairs (f, g) of holomorphic maps from M to CP^2 and $(CP^2)^\wedge$, respectively, and f and g are dual to each other. Let ∂f have area $m\pi$ and mapping degree d. One easily deduces the relation (cf. [4])

$$m = \deg(f) + \deg(g), \quad d = \deg(f) - \deg(g).$$

It is clear now that the moduli space of all branched superminimal immersions with fixed genus g, area $m\pi$ and degree d is identical with the space of all pairs of holomorphic maps f and g from any compact Riemann surface M of genus g (which varies in M_g), where f and g are dual to each other and both of them have fixed degrees.

On the other hand, due to the existence of Riemann surfaces with automorphisms, the set of all $M \in M_g$ does not form a "family" over M_g (in the sense that will be explained in (I) below, cf. [8]). We will therefore work with the Teichmüller moduli space T_g over which there is a universal analytic family of compact Riemann surfaces. For reader's convenience, some results from the theory of families of holomorphic maps presented in [9] is summarized in the following three points:

(I) Let $t \in T_g$ and let M_t be the corresponding compact Riemann surface. Then $\{M_t\}_{t \in T_g}$ forms an analytic family of compact Riemann surfaces. More precisely, the disjoint union $X_g = \bigcup_{t \in T_g} M_t$ has the structure of an analytic space and the natural projection $\pi: X_g \to T_g$ is a proper, smooth analytic map. Furthermore, X_g has an open covering of the form $U_i \times V_i$, where U_i is an open set of C and V_i is an open set of T_g, such that to each $(p, t) \in U_i \times V_i$, we have $\pi(p, t) = t$. The family $\{M_t\}_{t \in T_g}$ is universal in the sense that any other family $\{M_{t'}\}_{t' \in T'}$ is induced by an analytic map from T' to T.

(II) Let $\mathrm{Hol}_d(M_t, CP^2)$ be the set of all nondegenerate holomorphic maps of degree d from M_t to CP^2. Let $F_{g,d} = \bigcup_{t \in T_g} \mathrm{Hol}_d(M_t, CP^2)$. Then $F_{g,d}$ is an analytic space, which parametrizes a universal family of holomorphic maps $\{f_r\}_{r \in F_{g,d}}$. More precisely, if $b: F_{g,d} \to T_g$ is the natural projection, then b will induce a family of Riemann surfaces over $F_{g,d}$, denoted $\{M_r\}_{r \in F_{g,d}}$, whose total space is $b^*(X_g) = \bigcup_{r \in F_{g,d}} M_r$. The important point is that there exists an analytic map $\boldsymbol{F}_{g,d}: b^*(X_g) \to CP^2$ such that when restricted to M_r, $\boldsymbol{F}_{g,d}$ is just f_r, and f_r is just $r \in F_{g,d}$ regarded as a map from M_r to CP^2.

(III) Since $b^*(X_g)$ is a family, locally it is a product as explained in (I) so that we can use a pair (p, r) to specify the point of $b^*(X_g)$ in a small neighborhood. Here p varies in an open set of C, and r varies in an open set of $F_{g,d}$. For any fixed r, p is regarded as a local coordinate of M_r. With these, we can summarize the above discription of a "family of holomorphic maps" in the equation

$$\boldsymbol{F}_{g,d}(p, r) = f_r(p). \tag{6.1}$$

For our purpose, we must consider pairs of holomorphic maps f and g which are dual to each other and have fixed degrees. Consider the families $\{f_r\}_{r \in F_{g,d_1}}$ and $\{g_s\}_{s \in F_{g,d_2}}$ of holomorphic maps from $\{M_t\}_{t \in T_g}$ to CP^2 and $(CP^2)^\wedge$, respectively.

DEFINITION 6.1. Define

$$F_{g,d_1,d_2}=\{(r,s)\in F_{g,d_1}\times F_{g,d_2}\,|\,b(r)=b(s), f_r \text{ and } g_s, \text{ are dual to each other.}\}.$$

The condition $b(r)=b(s)\in T_g$ means f_r and g_s have the same domain $M_{b(r)}=M_{b(s)}$.

LEMMA 6.1. *F_{g,d_1,d_2} is an analytic subspace of $F_{g,d_1}\times F_{g,d_2}$.*

PROOF. F_{g,d_1,d_2} is defined to be a subset of $F_{g,d_1}\times F_{g,d_2}$ by two conditions. The first condition $b(r)=b(s)$ is clearly a closed condition. We will now prove that the second condition that f_r and g_s are dual to each other is also closed as well. Given $(r_0, s_0)\in F_{g,d_1}\times F_{g,d_2}$, let $b(r_0)=b(s_0)=t_0$ so that the maps f_r and g_s have the same domain M_{t_0}. Choose a point $p_0\in M_{t_0}$ such that p_0 is a smooth point of all three functions $f_r, \hat{f}_r$ and g_r, where $\hat{f}_r$ is the dual curve of f_r. From (6.1), the families $\{f_r\}$ and $\{g_s\}$ have local representation $f_r(p)$, $g_s(p)$ around (p_0, r_0) and (p_0, s_0) which are analytic when regarded as functions of (p,r) and (p,s), respectively. If we take (p,r) and (p,s) to be in small enough neighborhoods O_1 and O_2 of (p_0, r_0) and (p_0, s_0), respectively, then for each fixed r and s, the maps $f_r(p)$, $\hat{f}_r(p)$ and $g_r(p)$ will be smooth.

Take a nowhere vanishing holomorphic lift $\tilde{\boldsymbol{F}}(p,r)$ of $\boldsymbol{F}_{g,d}(p,r)$ around (p_0, r_0). By (6.1), we can define $\tilde{f}_r(p)=\tilde{\boldsymbol{F}}(p,r)$. For a fixed r, $\tilde{f}_r(p)$ gives a local lift of f_r around p_0, and $\tilde{f}_r(p)\wedge\tilde{f}'_r(p)$ is a nowhere vanishing local holomorphic lift of the dual curve $\hat{f}_r$, since $\hat{f}_r$ is smooth locally. We can define a similar local lift $\tilde{g}_s(p)$ for g_s. The condition that f_r and g_s are dual to each other is equivalent to the fact that $\hat{f}_r(p)=g_s(p)$ holds in a small open set $U\in M_{b(r)}=M_{b(s)}$. In terms of the local lifts introduced above, this is equivalent to saying that

$$(\tilde{f}_r(p)\wedge\tilde{f}'_r(p))\wedge\tilde{g}_s(p)=0 \tag{6.2}$$

holds in U, which is clearly a closed condition. Here, we use the convention $e_0\wedge e_1=e_2$, $e_1\wedge e_2=e_0$, $e_2\wedge e_0=e_1$ with respect to the orthonormal basis e_0, e_1, e_2. q.e.d.

To study the generic rigidity in the moduli space of superminimal surfaces naturally identified with F_{g,d_1,d_2}, we will first prove that the divisor $N_{f_r}\in M_r\times\bar{M}_r$ is prime for a generic r. For this purpose, we will realize all N_{f_r} as curves in a common projective space and apply the concept of Chow variety introduced in Section 2. According to a theorem in the theory of Teichmüler space (cf. [8]), the family $\{M_t\}_{t\in T_g}$ of compact Riemann surfaces can be simultaneously embedded into the common space $\boldsymbol{CP}^{5g-6}$. (When $g=1$, $\boldsymbol{CP}^{5g-6}$ should be replaced by $\boldsymbol{CP}^2$.) More precisely, there is an analytic map $\boldsymbol{E}_g\colon X_g\to\boldsymbol{CP}^{5g-6}$ such that when restricted to each M_t, the map $\boldsymbol{E}_g$ is an embedding $e_t\colon M_t\to\boldsymbol{CP}^{5g-6}$ of degree $6g-6$. In this way, we can always identify M_r with $e_r(M_r)\subset\boldsymbol{CP}^{5g-6}$. Recall that the family $\{M_r\}_{r\in F_{g,d}}$ is induced from the family $\{M_t\}_{t\in T_g}$ through an analytic map $b\colon F_{g,d}\to T_g$. It is clear that we thus have an induced analytic map

$$\boldsymbol{E}_{g,d}\colon b^*(X_g)\to\boldsymbol{CP}^{5g-6}$$

such that when restricted to each M_r, the map $\boldsymbol{E}_{g,d}$ is an embedding $e_r\colon M_r \to CP^{5g-6}$. We can then consider the map

$$\boldsymbol{E}_{g,d} \times \boldsymbol{F}_{g,d}\colon b^*(X_g) \to CP^{5g-6} \times CP^2$$

whose restriction to $\boldsymbol{M}_r$ is an embedding $e_r \times f_r\colon M_r \to CP^{5g-6} \times CP^2$. Identifying M_r with its image $e_r(M_r) \subset CP^{5g-6}$, we see that the image $e_r \times f_r(M_r) \subset CP^{5g-6} \times CP^2$ can be thought of as the graph of the map f_r.

Recall from (3.1) that N_{f_r} is the pullback of $\boldsymbol{S}_0$ through the sequence

$$M_r \times \bar{M}_r \xrightarrow{f_r \times \bar{f}_r} CP^2 \times \overline{CP}^2 \xrightarrow{\text{Segre}} CP^8 .$$

We will consider the family of complex surfaces $\{M_t \times \bar{M}_t\}_{t \in T_g}$ and the corresponding family of analytic maps $\{f_r \times \bar{f}_r \mid r \in F_{g,d}\}$; notice that $\bar{M}_t$ and $\bar{f}_r$ vary anti-holomorphically with respect to t and r, we must consequently regard the two families as real analytic families. To each fixed $r \in F_{g,d}$, there is an embedding

$$e_r \times \bar{e}_r\colon M_r \times \bar{M}_r \to CP^{5g-6} \times \overline{CP}^{5g-6}$$

and an analytic map $f_r \times \bar{f}_r\colon M_r \times \bar{M}_r \to CP^2 \times \overline{CP}^2$ so that we can consider

$$M_r \times \bar{M}_r \xrightarrow{e_r \times \bar{e}_r \times f_r \times \bar{f}_r} CP^{5g-6} \times \overline{CP}^{5g-6} \times CP^2 \times \overline{CP}^2$$

$$\xrightarrow{\text{identity} \times \text{Segre}} CP^{5g-6} \times \overline{CP}^{5g-6} \times CP^8.$$

DEFINITION 6.2. Define G_r to be the image of the above map, and define $\boldsymbol{R}_0$ to be $CP^{5g-6} \times \overline{CP}^{5g-6} \times \boldsymbol{S}_0 \subset CP^{5g-6} \times \overline{CP}^{5g-6} \times CP^8$.

G_r can be thought of as the graph of $f_r \times \bar{f}_r$, which is a surface of degree $2(6g-6+d)^2$, and $G_r \cap \boldsymbol{R}_0$ can be thought of as N_{f_r}. With this interpretation of N_{f_r}, we have:

LEMMA 6.2. *Let $W_{g,d} = \{r \in F_{g,d} \mid N_{f_r} = G_r \cap \boldsymbol{R}_0$ is prime$\}$. Then $W_{g,d}$ is Zariski open dense in $F_{g,d}$ considered as a real analytic space.*

PROOF. Consider the sequence of maps $F_{g,d} \to \boldsymbol{C}_1 \to \boldsymbol{C}_2$ defined by $r \mapsto G_r$ and $G \mapsto G \cap \boldsymbol{R}_0$, where $\boldsymbol{C}_1$ is the variety consisting of all the surfaces G of degree $2(6g-6+d)^2$ in $CP^{5g-6} \times \overline{CP}^{5g-6} \times CP^8 \subset CP^{9(5g-5)^2-1}$ which does not lie entirely in $\boldsymbol{R}_0$, and $\boldsymbol{C}_2$ is the variety consisting of all curves of degree $2d(6g-6+d)$ in $CP^{5g-6} \times \overline{CP}^{5g-6} \times CP^8 \subset CP^{9(5g-5)^2-1}$. The map $r \mapsto G_r$ is real analytic, and the map $G \mapsto G \cap \boldsymbol{R}_0$ is complex analytic because G and $\boldsymbol{R}_0$ intersect properly in $CP^{5g-6} \times \overline{CP}^{5g-6} \times CP^8$ (cf. [6]). Here both $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are subvarieties of appropriate Chow varieties. Note that a generic element in a Chow variety is irreducible; hence by the same argument as in the proof of Lemma 5.5, our lemma will be proved if in each irreducible component of $F_{g,d}$, we can find at least one r such that $N_{f_r} = G_r \cap \boldsymbol{R}_0$ is prime. But this will follow from Lemma 5.5 because for any projective transformation $A: CP^2 \to CP^2$, the map Af_r is still in $F_{g,d}$ and lies in the same irreducible component of $F_{g,d}$ as f_r. That Af_r and f_r lie in the same irreducible component of $F_{g,d}$ follows from the fact that the connected group $PGL(3, C)$ acts continuously on $F_{g,d}$ by $f_r \mapsto Af_r$, which therefore leaves the irreducible components of $F_{g,d}$ invariant. q.e.d.

LEMMA 6.3. *Let* $W_{g,d_1,d_2} = \{(f_r, g_s) \in F_{g,d_1,d_2} \mid N_{f_r}$ *and* N_{g_s} *are prime*$\}$. *Then* W_{g,d_1,d_2} *is Zariski open dense in* F_{g,d_1,d_2} *regarded as a real analytic space.*

PROOF. Note that for each $(f_r, g_s) \in F_{g,d_1,d_2}$ and each projective transformation $A: CP^2 \to CP^2$, $(Af_r, \hat{A}g_s)$ is still in F_{g,d_1,d_2} and lie in the same irreducible component as (f_r, g_s). The same argument as in the proof of Lemma 6.2 holds. q.e.d.

Recall that F_{g,d_1,d_2} can be regarded as an analytic family of superminimal surfaces of degree $d = d_1 - d_2$ and area $m\pi = (d_1 + d_2)\pi$ in CP^2. The group $PGL(3, C)$ acts on F_{g,d_1,d_2} by $(f, \tilde{f}) \mapsto (Af, \widetilde{Af})$, which induces an analytic fibration (cf. [9])

$$PGL(3, C) \to F_{g,d_1,d_2} \to F_{g,d_1,d_2}/PGL(3, C) .$$

In terms of this fibration, Theorom 2 can be interpreted as fiberwise generic rigidity. By Lemma 6.3 and the argument in the proof of Theorem 2, we can immediately generalize the statement of Theorem 2 to generic rigidity on the total space F_{g,d_1,d_2}, namely, each $r \in W_{g,d_1,d_2}$ gives a rigid superminimal surface, and being Zariski open dense, W_{g,d_1,d_2} intersects each fiber of F_{g,d_1,d_2} in a Zariski open dense set.

The action of the Teichmüller modular group Γ_g on T_g naturally induces an action of Γ_g on F_{g,d_1,d_2}. The quotient space $F_{g,d_1,d_2}/\Gamma_g$ is actually the set of branched superminimal immersions of genus g, degree d and area $m\pi$. It is clear that W_{g,d_1,d_2} in Lemma 6.3 is invariant under Γ_g. Hence $W_{g,d_1,d_2}/\Gamma_g$ is Zariski open dense in the real analytic space $F_{g,d_1,d_2}/\Gamma_g$. We have thus arrived at the following:

THEOREM 3. *Superminimal surfaces of genus* g, *degree* d *and area* $m\pi$ *in* CP^2 *form an analytic variety* $F_{g,d_1,d_2}/\Gamma_g$ *with a natural action of* $PGL(3, C)$, *on which there is a Zariski open dense set* $W_{g,d_1,d_2}/\Gamma_g$ *consisting of rigid superminimal surfaces. Moreover, this Zariski open dense set intersects each orbit of the* $PGL(3,C)$*-action in a Zariski open dense set of the orbit.*

It should be mentioned that the Brill-Noether theory of algebraic curves guarantees the existence of nonempty F_{g,d_1,d_2} under general conditions on g and d_1 (cf. [1]).

RAMARK. The proof of Lemma 2 in [2] is incomplete, where Lemma 3 does not yield an example with irreducible $\|\psi\|^2$ and $\|\psi \wedge \psi'\|^2$ as long as $\deg \psi \geq 4$. To construct an example for any degree, one can either quote Theorem 2 of this paper, or exhibit a concrete irreducible polynomial as follows. Consider the one-parameter family $f_t(z, \bar{z}) = 2 + z + tz + tz\bar{z} + z^n\bar{z}^n$. When $t = 0$, $f_0(z, \bar{z}) = 2 + z + z^n\bar{z}^n$, which is irreducible in z and $\bar{z}$. Consequently there is a small real $t_0 > 0$ such that $f_{t_0}(z, \bar{z})$ is irreducible, because the set of irreducible polynomials in z and $\bar{z}$ is Zariski open among all polynomials in z and $\bar{z}$ of degree n. By changing $\bar{z}$ to $t_0\bar{z}$, $f_{t_0}(z, \bar{z})$ can be rewritten as $F=2+z+\bar{z}+z\bar{z}+t_0^{-n}z^n\bar{z}^n$. Set $\psi(z)=(1, 1+z, t_0^{-n/2}z^n)$. Then $\|\psi\|^2=F$, which is irreducible. Similarly by changing z to z^{-1} with respect to F, we see that the polynomial $G=t_0^{-n}+z^{n-1}\bar{z}^n+z^n\bar{z}^{n-1}+z^{n-1}\bar{z}^{n-1}+2z^n\bar{z}^n$ is irreducible. In particular, setting $\mu=(t_0^{-n/2}, z^{n-1}+z^n, z^n)$, we have $\|\mu\|^2=G$. Let $\tau=(1, z, z^n)$. It is clear that $\tau\wedge\tau'=((n-1)z^n, -nz^{n-1}, 1)$ is linearly equivalent to μ, that is, $\mu=A\cdot(\tau\wedge\tau')$ with $A\in GL(3, \boldsymbol{C})$. Here we identify (e_0, e_1, e_2) and $(e_1\wedge e_2, e_2\wedge e_0, e_0\wedge e_1)$ as the standard basis of $\boldsymbol{C}^3$ and $\bigwedge^2\boldsymbol{C}^3$, respectively. Then $\psi=(A^{-1})^t\tau$ satisfies $\psi\wedge\psi'=\mu$. Hence $\|\psi\wedge\psi'\|^2=G$ is irreducible.

Of course, this argument is just a concrete realization of Theorem 2 of this paper in the sphere case in $\boldsymbol{CP}^2$. Note also that Lemma 3 in [2] is precisely Theorem 2 in the case of plane cubic cuspidal curves.

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