HOMEWORK 8, MATH 233
DUE MONDAY, OCTOBER 28, 2002

Each of the seven problems is worth 1 point for a total of 7 points.

(1) Suppose \( u(x, t) \) is a twice continuously differentiable function satisfying the vibrating string equation

\[
\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2}
\]

where \( a \) is a positive scale constant. Define new variables \( r \) and \( s \) by \( r = x + at \), \( s = x - at \). Equivalently, \( x = (r + s)/2 \), \( t = (r - s)/2a \). Use the chain rule to show that when \( u \) is regarded as a function of \( r \) and \( s \), it satisfies

\[
\frac{\partial^2 u}{\partial r \partial s} = 0
\]

Then deduce that \( u \) has the form \( u = f(r) + g(s) = f(x + at) + g(x - at) \) for some functions \( f \) and \( g \). As discussed in class, the interpretation of this is that every solution to the vibrating string can be thought of as the superposition of a wave traveling to the left and a wave traveling to the right.

(2) In thermodynamics and physical chemistry, it is assumed that every gas satisfies an equation of state \( F(P, V, T) = 0 \) where \( P, V, T \) are the pressure, volume, and temperature of the gas. One can take experimental data to determine the extent to which \( PV/T \) stays constant— if so, the equation of state takes the form \( PV = (\text{constant})T \) and the gas is called ideal. Many gases are approximately ideal, others are far from being ideal. In general, \( F \) isn’t explicitly known but it’s assumed to be continuously differentiable with none of its partial derivatives ever 0. The implicit function theorem then guarantees that the equation of state can be solved for any one of the three variables as a continuously differential function of the other two. It’s fairly easy to set up laboratory experiments giving approximations from data for \( \partial P/\partial V \) [let the gas expand but hold the temperature constant and see how fast the pressure drops] and \( \partial P/\partial T \) [apply heat to raise the temperature but keep the gas in a container of fixed volume and see how fast the pressure increases]. It’s not so easy to set up experiments giving data approximations for the remaining 4 partial derivatives \( \partial V/\partial P, \partial V/\partial T, \partial T/\partial V, \partial T/\partial P \). Fortunately, it’s not necessary to do so. Use the implicit differentiation formulas discussed in class to give formulas expressing each of these last four partial derivatives in terms of the first two.

(3) Do #36, page 789, §11.4. First compute by hand the equation of the tangent plane to the surface at the specified point. Then use Matlab to plot both the surface and the tangent plane on the same plot.

(4) Use Lagrange multipliers to find the maximum and minimum values of the function \( f(x, y, z) = x - 3y + z \) subject to the constraint \( x^2 + 2y^2 + z^2 = 4 \).
(5) (#18, page 828, §11.8).
(a) Find the extreme values of \( f(x, y) = 2x^2 + 3y^2 - 4x - 5 \) on the region \( x^2 + y^2 \leq 16 \).
(b) Use Matlab to plot \( z = f(x, y) \) over this region. Use cylindrical coordinates. Adjust the view. On the graph mark by hand the location of the maximum and minimum points.

(6) Do #2 on page 836, Chapter 11 Focus.

(7) Do #52 on page 835, Review Problems. As indicated, first find the local max/min and saddle points by hand, then use Matlab to graph the function over a domain containing all of these points and use Rotate 3d to get a good view.

Extra Credit Problems

#386, p. 828
#8, p. 837
1. Given $u(x)$, two continuously differentiable functions satisfy $u_t = a^2 u_{xx}$ for $a > 0$.

   (a) With $x = t$ and $t = x^2$, so $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$.

   By the chain rule,
   \[
   \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.
   \]

   \[2 \frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0.
   \]

   \[u(x, t) = f(x) + g(t).
   \]

   (b) If $f(0) = g(0) = 0$, then
   \[u(x, t) = f(x) + g(t) \quad \Rightarrow \quad u(x, t) = f(x) + g(t).
   \]

Taking $t = \frac{x}{a}$, the result is $u(x, t) = f(x) + g(t)$ both for $x > 0$ and $t > 0$.

For each fixed value of $x$ and letting $u = u(x, t)$, and
\[u_t = f(x) + g(t) \quad \Rightarrow \quad u_t = f(x) + g(t).
\]

Reverting back to the variables $x$ and $t$,
\[u(x, t) = f(x + at) + g(x - ct)\]
By the assumptions on \( F \), the equation \( F(P, V, T) = 0 \) can be solved for any one of the 3 variables as a differentiable function of the other 2 variables.

\[
\frac{\partial P}{\partial V} \neq \frac{\partial F/\partial V}{\partial F/\partial P} \quad \frac{1}{\partial V/\partial T} = \frac{1}{\partial P/\partial T}.
\]

Similarly,

\[
\frac{\partial V}{\partial T} = \frac{1}{\partial T/\partial P} \frac{\partial P}{\partial V} \quad \frac{\partial V}{\partial T} = \frac{\partial V}{\partial P} \frac{\partial P}{\partial T}.
\]

In this way, knowledge of \( \partial P/\partial T \) and \( \partial P/\partial V \) completely determines the values of \( \partial T/\partial P, \partial V/\partial T, \partial T/\partial V \).
\( r^3(u,v) = \langle uv, u^2, v^2 \rangle \)

with \( r^3(0,0) = \langle 0,0,0 \rangle \)

\( r^3_u(0,0) = \langle 0,1,0 \rangle = \vec{j} \)

\( r^3_v(0,0) = \langle 0,0,1 \rangle = \vec{k} \)

So \( r^3_u(0,0) \times r^3_v(0,0) = \vec{j} \times \vec{k} = \vec{z} = \langle 1,0,0 \rangle \)

is a normal vector to the tangent plane of the surface at \((0,0,0)\)

and the equation of the tangent plane is \([0=x,y,z] = \vec{z} = \langle 1,0,0 \rangle \)

[the tangent plane is the \(yz\) plane]

Remarks: We can then parametrize the tangent plane by \( r^3(y,z) = \langle 0,y,z \rangle \)

or get Matlab to sketch it as the plane containing the 4 points \((0,-2,-2), (0,-2,2), (0,2,-2), \)

and \((0,2,2)\)

In a Matlab sketch of the surface we can take \(u\) and \(v\) to vary from \(-1\) to \(1\)

in increments of \(0.05\) with

\[ x = u \, v \]

\[ y = u^2 \, v \]

\[ z = v^2 \, u \]
ENW Plot: the surface, $ue$, $ve'$, and tangent plane.
Find the maximum and minimum values of
\[ f(x, y, z) = x - 3y + z \] subject to the constraint
\[ g(x, y, z) = x^2 + 2y^2 + z^2 = 4 \]
\(< -3, 1 > = \nabla f = \lambda \nabla g = \lambda \langle x, 2y, 2z \rangle \]
\[ \lambda \left( \begin{array}{c} x \\ 2y \\ 2z \end{array} \right) = \left( \begin{array}{c} x \\ 2y \\ 2z \end{array} \right) \]
and
\[ x = 2 \]
\[ y = -\frac{3}{2} \]

with \[ y = x^2 + 2y^2 + z^2 = \frac{17}{4} \]

Then \[ x = \pm \frac{4}{\sqrt{17}} \] and we have
the two points \( \left( \frac{4}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right) \)
and \( \left( -\frac{4}{\sqrt{17}}, \frac{3}{\sqrt{17}}, -\frac{4}{\sqrt{17}} \right) \)
satisfying the Lagrange multiplier equations.

Therefore the maximum value of \( f \) must be
\[ f\left( \frac{4}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right) = \frac{1}{\sqrt{17}} \left( 8 + (-18) \right) = \frac{26}{\sqrt{17}} \]

and the minimum value is
\[ f\left( -\frac{4}{\sqrt{17}}, \frac{6}{\sqrt{17}}, -\frac{4}{\sqrt{17}} \right) = \frac{-26}{\sqrt{17}} \]
Find the extreme values \( f(x, y) = 3x^2 - 5 \)
on the region \( D \): \( x^2 + y^2 \leq 16 \)

**Critical Points**
- In the interior of \( D \):
  \[ f_x = 6x, \quad f_y = 0 \]
  - \( x = 0 \)
  - \( y = 0 \)

**Lagrange Multipliers**
- On the boundary of \( D \):
  \[ (x, y) \in \partial D \]

For \( y = 0 \), \( 2x^2 = 16 \) and \( k = 4 \)

with

\[ f(4, 0) = 2(16) = 4(4) = 16 \]

For \( y \neq 0 \):
- \( \frac{f_y}{f_x} = \frac{2}{2} = 1 \)

with \( y^2 = 6x \)

Thus \( f(x, \pm \sqrt{12}x) = 3(12) = 36 \)

**Maxima**
- \( \max f = 36 \) achieved at \((\pm \sqrt{12}, 0)\)

**Minima**
- \( \min f = -7 \) achieved at \((1, 0)\)
Plot of the surface \( z = 2x^2 + 3y^2 - 4x - 5 \) over the disk \( x^2 + y^2 \leq 1 \).
(a) The level curves of \( C(x, y) = e^{x^2 + 2y^2} \) are ellipses. Let \( k = C \) be constant.

With \( C = e^{-k/16} \) and the shock path traced inward toward the \( x \) axis, the greatest and face normal to each tangent intersects.

(b) Integrate \( \langle x(t), y(t) \rangle \) to parameterize the shock path.

\[
\frac{dx}{dt} \quad \frac{dy}{dt} \quad \text{must be a multiple of } \nabla C(x(t), y(t))
\]

\[
\begin{align*}
&\frac{dx}{dy} \quad \frac{dy}{dx} = 2x y \\
\text{or} \\
&\frac{dy}{dx} = 2x \\
\end{align*}
\]

\[
\ln y = 2 \ln x + \text{constant}
\]

\[
\text{The path goes through } (x_0, y_0) \text{ with the constant } \ln y_0 = 2 \ln x_0
\]

and \( \ln y/y_0 = 2 \ln x/x_0 \)

\[
\text{or} \quad y/y_0 = x^2/x_0^2
\]

Equation of a parabola
\[ f(x, y) = x^3 - 6xy + 8y^3 \]

**Critical Points:**

1. \[ 0 = f_x = 3x^2 - 6y = 3(x^2 - 2y) \]
2. \[ 0 = f_y = -6x + 24y^2 = 6(4y^2 - x) \]

So \[ x^2 = 2y \] and \[ 4y^2 = x \]

Then \[ 2y = (4y^2)^2 = 16y^4 \]

which has solutions \[ y = 0 \]

and \[ y = (\frac{1}{8})^{\frac{1}{3}} = \frac{1}{2} \]

For \( y = 0 \), \( x = 4(0^2) = 0 \)

For \( y = \frac{1}{2} \), \( x = 4(\frac{1}{2})^2 = 1 \)

the critical points of \( f \) are \( (0, 0) \) and \( (1, \frac{1}{2}) \)

We have \[ f_{xx} = 6x \]

\[ f_{xy} = -6 \]

\[ f_{yy} = 48y \]

At \( (0, 0) \), \[ D = f_{xx}f_{yy} - f_{xy}^2 = 36 > 0 \]

so \( (0, 0) \) is a saddle point

At \( (1, \frac{1}{2}) \), \[ D = f_{xx}f_{yy} - f_{xy}^2 = (6)(24) - 36 > 0 \]

with \( f_{xx} \) and \( f_{yy} > 0 \) so

\( (1, \frac{1}{2}) \) is a local minimum point
ENW, Plot of the surface $z = x^3 - 6xy + 8y^3$. 

Local minimum

Saddle point
Extra Credit  # 38b, p. 828

We want to find the highest and lowest points on the ellipse arising from the intersection of the plane \(4x - 3y + 8z = 5\) and the cone \(z^2 = x^2 + y^2\).

Thus we want to find the maximum and minimum values of \(f(x, y, z) = z\) subject to the constraints \(g(x, y, z) = 4x - 3y + 8z = 5\) and \(h(x, y, z) = z^2 - x^2 - y^2 = 0\).

These will occur at points where \(\nabla f = \lambda \nabla g + \mu \nabla h\), i.e. \(\langle 0, 0, 1 \rangle = \lambda \langle 4, -3, 8 \rangle + \mu \langle -x, -y, 2 \rangle\).

Since \(\langle 0, 0, 1 \rangle\) isn't a multiple of \(\langle 4, -3, 8 \rangle\), we must have \(\mu \neq 0\) and then
\[
4\lambda = (2\mu)x \\
-3\lambda = (2\mu)(-y)
\]

Which gives \(\frac{2}{\lambda} = \frac{x}{2} = -\frac{y}{3}\) or \(y = -\frac{3x}{4}\).

Then \(z^2 = x^2 + y^2 = x^2 + \frac{9x^2}{16} = \frac{16x^2}{16}\). So \(\pm z = \frac{5x}{4}\).

And \(5 = 4x - 3y + 8z = 4x + \frac{9x}{4} + 8z = \frac{25x}{4} + 8z\).

With \(z = \frac{5x}{4}\), \(5z = \frac{25x}{4}\) and
\[
5 = 5z + 8z = 13z \Rightarrow 5 = \frac{25x}{4} + 8z
\]

So \(z = \frac{5}{13}, x = \frac{4}{13}, y = -\frac{3}{13}\).

With \(z = -\frac{5x}{4}\), \(5z = -\frac{25x}{4}\) and
\[
5 = -5z + 8z = 3z
\]

So \(z = \frac{5}{3}, x = -\frac{4}{3}, y = 1\).

The first point gives the smallest \(z\) and the second point gives the largest \(z\).
Among all planes tangent to \( x^2 y^2 z^2 = 1 \), find the ones farthest from the origin.

At \((x_0, y_0, z_0)\), \( \nabla F(x_0, y_0, z_0) = \langle y_0 x_0^2, 2 x_0 y_0 z_0, 2 x_0 y_0 z_0 \rangle \) is a normal vector along \( x_0 y_0 z_0 = 1 \), \( \langle \frac{1}{x_0}, \frac{2}{y_0}, \frac{2}{z_0} \rangle \) is a normal vector and the equation of the tangent plane at \((x_0, y_0, z_0)\) is
\[
(x - x_0) \frac{x}{x_0} + (y - y_0) \frac{y}{y_0} + (z - z_0) \frac{z}{z_0} = 0.
\]
The distance of this plane from the origin is
\[
\left| \frac{-x_0 \frac{x}{x_0} - y_0 \frac{y}{y_0} - z_0 \frac{z}{z_0}}{\sqrt{\frac{1}{x_0^2} + \frac{4}{y_0^2} + \frac{4}{z_0^2}}} \right| = \frac{S}{\sqrt{\frac{1}{x_0^2} + \frac{4}{y_0^2} + \frac{4}{z_0^2}}}.
\]

Maximize this distance by minimizing
\[
G(x_0, y_0, z_0) = \frac{1}{x_0^2} + \frac{4}{y_0^2} + \frac{4}{z_0^2}
\]
subject to the constraint \( F(x_0, y_0, z_0) = 1 \).

Then \( \nabla G(x_0, y_0, z_0) = \lambda \nabla F(x_0, y_0, z_0) \)
gives \( \langle -\frac{2}{x_0^3}, -\frac{4}{y_0^3}, -\frac{4}{z_0^3} \rangle = \lambda \langle \frac{y_0^2}{x_0}, \frac{2x_0z_0}{y_0}, \frac{2x_0z_0}{y_0} \rangle \)
or \( \frac{-2}{x_0^3} = \frac{x_0^3}{x_0^3} = \frac{y_0^3}{y_0^3} = \frac{z_0^3}{z_0^3} \).

Thus \( x_0^2 = y_0^2 = z_0^2 \)
with \( 1 = x_0 y_0 z_0 \Rightarrow x_0 = \pm 1, y_0 = \pm 1, z_0 = \pm 1 \).

The maximal distance is then \( \frac{S}{\sqrt{1 + 4 + 4}} = \frac{S}{3} \)
and the 4 planes achieving this maximal distance have the equations (4) using the value \( 8 \times 0, 0, 0 \) we have obtained.