HOMEWORK 11, MATH 233
DUE THURSDAY, NOVEMBER 20, 2003

Each problem is worth one point for a total of 6 points.

(1) Do #30 of 12.8.
(2) Do #12 of 12.9.
(3) Do #20 of 12.9.
(4) Do #22 of 12.9.
(5) Do #42 on page 912 of chapter 12 review.
(6) Do #50 on page 913 of chapter 12 review.
(1) 12.8, #30.

Note \[
\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy
\]

\[= \iiint_E (x^2 + y^2 + z^2) \, dV\]

Where \(E\) is the solid region in 3-d that lies above the cone \(z = \sqrt{x^2 + y^2}\), below the (hemisphere) sphere \(z = \sqrt{18 - x^2 - y^2}\) over the region \(D\) in the xy-plane.

In spherical coordinates, \(E = \{(\rho, \phi, \theta) : 0 \leq \rho \leq 3\sqrt{2}, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \theta \leq \frac{\pi}{2}\}\).

Use spherical coordinates,

\[
\iiint_E (x^2 + y^2 + z^2) \, dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{\frac{3\sqrt{2}}{2}}} \rho^2 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \rho^5 \sin \phi \, d\rho \, d\phi
\]

\[= \frac{4\sqrt{2} - 6}{5} \pi
\]
The line in the xy-plane passing through \((5,1)\) and \((3,-2)\) has equation \(3x - 2y - 13 = 0\).

Since \(x = 2u + 3v\), \(y = 3u - 2v\), we have \(13v - 13 = 0\), or \(v = 1\).

So \(T^{-1}\) transforms the above line to the line \(v=1\) in the uv-plane.

Similarly \(T^{-1}\) transforms the line in xy-plane through \((0,0)\) and \((3,-2)\) to the line \(u=0\) in uv-plane, the line in xy-plane through \((0,0)\) and \((2,3)\) to the line \(v=0\) in uv-plane, the line in xy-plane through \((2,3)\) and \((5,1)\) to the line \(u=1\) in uv-plane.

\[
\begin{vmatrix}
2 & 3 \\
3 & -2
\end{vmatrix} = -13
\]

\[
\iint_{R} (x+y)\,dA = \int_{S} \int_{S} (2u+3v+3u-2v) \,dudv = 13 \int_{0}^{1} \int_{0}^{1} (5u + v) \,dudv = 13 \int_{0}^{1} \frac{5}{2}u + v \,dv = 39
\]
Let $\mathbf{u} = x + 2y$, $\mathbf{v} = x - y$.

\begin{align*}
\text{Solve for } x, y \text{ in terms of } \mathbf{u}, \mathbf{v}.
\end{align*}

\begin{align*}
x &= \frac{\mathbf{u} + 2\mathbf{v}}{3}, \\
y &= \frac{\mathbf{u} - \mathbf{v}}{3}
\end{align*}

\begin{align*}
\frac{\partial (x, y)}{\partial (\mathbf{u}, \mathbf{v})} &= \begin{vmatrix}
\frac{1}{3} & \frac{2}{3} \\
\frac{1}{3} & -\frac{1}{3}
\end{vmatrix} = -\frac{1}{3}
\end{align*}

Note:

- $\mathbf{u} = 0 \quad \rightarrow \quad x + 2y = 0$
- $\mathbf{u} = 2 \quad \rightarrow \quad x + 2y = 2$
- $\mathbf{v} = 0 \quad \rightarrow \quad x = y$
- $\mathbf{v} = 1 \quad \rightarrow \quad y = x - 1$

\[
\int \int_{R} \frac{x + 2y}{\cos(x-y)} \, dA = \int_{0}^{1} \int_{0}^{2} \frac{\mathbf{u}}{\cos(\mathbf{v})} \cdot \frac{1}{3} \, d\mathbf{u} \, d\mathbf{v}
\]

\[
= \int_{0}^{1} \left( \int_{0}^{2} \frac{1}{3 \cos(\mathbf{v})} \cdot \frac{1}{2} \mathbf{u}^{2} \right)_{\mathbf{u}=0}^{\mathbf{u}=2} \, d\mathbf{v}
\]

\[
= \int_{0}^{1} \frac{2}{3} \cdot \frac{1}{\cos(\mathbf{v})} \, d\mathbf{v} = \frac{2}{3} \int_{0}^{1} \sec(\mathbf{v}) \, d\mathbf{v}
\]

\[
= \frac{2}{3} \ln \left| \sec(\mathbf{v}) + \tan(\mathbf{v}) \right|_{\mathbf{v}=1}^{\mathbf{v}=0}
\]

\[
= \frac{2}{3} \ln \left( \sec(1) + \tan(1) \right)
\]
Let \( x = \frac{1}{3} r \cos \theta \), \( y = \frac{1}{2} r \sin \theta \).

\[
3x = r \cos \theta, \quad 2y = r \sin \theta
\]

Then \( 9x^2 + 4y^2 = 1 \) gives \( r^2 = 1 \) or \( r = 1 \).

\[
\frac{\partial (x, y)}{\partial (r, \theta)} = \frac{1}{6} r
\]

\[
\int_{R} \sin (9x^2 + 4y^2) \, dA = \int_{0}^{\frac{\pi}{2}} \int_{0}^{1} \sin (r^2) \frac{1}{6} r \, dr \, d\theta
\]

\[
= \frac{\pi}{2} \left[ \frac{\sin (r^2)}{2} \right]_{0}^{1} \frac{1}{6} r \, dr
\]

\[
= \frac{\pi}{12} \left[ -\frac{1}{2} \cos (r^2) \right]_{r=0}^{1}
\]

\[
= \frac{\pi}{12} \left( -\frac{1}{2} \cos 1 + \frac{1}{2} \right)
\]

\[
= \frac{\pi}{24} (1 - \cos 1)
\]
\[ \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} (x^2 + y^2 + z^2)^2 \, dz \, dy \, dx \]

\[ = \iiint_{E} (x^2 + y^2 + z^2)^2 \, dV, \quad \text{where } E \text{ is the portion of} \]

the ball \( x^2 + y^2 + z^2 \leq 1 \) in

the first octant.

\[ = \frac{\pi}{2} \int_0^1 r^6 \, dr \int_0^{\frac{\pi}{2}} \sin \phi \, d\phi \]

\[ = \frac{\pi}{14} \]

In spherical coordinates,

\[ E: \quad 0 \leq r \leq 1, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2} \]
Under the transformation \( x = u^2, y = v^2, z = w^2 \), the region (in xyz space) bounded by the surface \( \sqrt{x} + \sqrt{y} + \sqrt{z} = 1 \) and the coordinate planes corresponds to the region (in uvw space) bounded by the plane \( u + v + w = 1 \) and the coordinate planes.

\[
\text{Vol}(R) = \iiint_R \, dv = \iiint_S \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| \, du \, dv \, dw
\]

\[
\frac{\partial (x, y, z)}{\partial (u, v, w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw
\]

\[
\iiint_S \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| \, du \, dv \, dw = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \, dw \, dv \, du
\]

\[
= \int_0^1 \int_0^{1-u} 4uv(1-u-v)^2 \, dv \, du
\]

\[
= \int_0^1 \int_0^{1-u} (4uv^3 - 8u(1-u)v^2 + 4u(1-u)^2v) \, dv \, du
\]

\[
= \int_0^1 \frac{u(1-u)^4}{3} \, du
\]

\[
= \frac{1}{90}.
\]