MATH 4111 HOMEWORK ASSIGNMENT #3
Due Thursday, September 29

1. #1(c), p. 61. Note that for $(\mathrm{E}_1, d_1) = (\mathrm{E}_2, d_2) = (\mathbb{R}, d_{|\cdot|})$, the associated metric d on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is what we called in class the uniform metric
$d_\infty(x, y) = \|x - y\|_\infty$. Similarly, for $n = k + l$ with k and l in $\mathbb{N}$, taking (E_1, d_1) =($\mathbb{R}^k, d_{\infty)}$ and (E_2, d_2) =($\mathbb{R}^l, d_{\infty)}$ and identifying $\mathbb{R}^n$ in the usual way with $\mathbb{R}^k \times \mathbb{R}^l$, the metric arising as in #1(c) is just the uniform metric d_∞ on $\mathbb{R}^n$. Also notice, that, in the notations of #1(c), $d^\sim((x_1, x_2), (y_1, y_2)) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2}$ defines a metric on $\mathrm{E}_1 \times \mathrm{E}_2$ which is equivalent to d. If (E_j, d_j) is a Euclidean space ($i.e.$ a real vector space equipped with an inner product norm), so is $(\mathrm{E}_1 \times \mathrm{E}_2, d^\sim)$ Don't bother to write up the details proving these two statements but you might want to check them for yourself.

2. #8, p. 61. Paraphrasing this exercise, if S is a countably infinite subset of a metric space (E,d) and there happens to be some way to count off the members of S which yields a sequence converging in (E,d) to some $p \in$ E,then, for every 1-1 function $k \mapsto p_k$ from $\mathbb{N}$ into S, $\lim_{k\to\infty} p_k = p$ in (E,d).

Recall that an infinite series $\sum_{k=0}^{\infty} a_k$ is said to converge if the partial sum sequence $(s_n)_{n\geq 0}$ converges where $s_n = \sum_{k=0}^{n} a_k$. Conversely, when a sequence $(s_n)_{n\geq 0}$ of real numbers or vectors ($e.g.,$ complex numbers) converges, it is the partial sum sequence of the infinite series defined by $a_0 = s_0$ and $a_k = s_k - s_{k-1}$ for $k \geq 1$. Although the ordering of the partial sum sequence is irrelevant to convergence, the ordering of the terms in an infinite series is highly relevant. We'll come back to this later.

3. #11, p. 62.

4. <u>Additional problem on summability methods</u>.

Whether or not an infinite series $\sum_{k=0}^{\infty} a_k$ of complex numbers converges in $\mathbb{C}$, the series is said to be Cesaro summable to $s \in \mathbb{C}$ if, with $s_n = \sum_{k=0}^{n} a_k$, $\lim_{n\to\infty}\left(\frac{\sum_{k=0}^{n} s_n}{n+1}\right) = s$. Alternatively, if the power series $f(t) = \sum_{k=0}^{\infty} t^k a_k$converges for each $t \in (0,1)$ and $s = \lim_{t\to 1} f(t)$ exists in $\mathbb{C}$, $\sum_{k=0}^{\infty} a_k$ is said to be Abel summable to s.

(i) Give an example of a non-converging infinite series and a number s for which the series is both Cesaro and Abel summable to s.

(ii) By #11 applied to the partial sums, if an infinite series converges to some $s \in \mathbb{C}$, the series is also Cesaro summable to s. Show that the series is also Abel summable to s; for this, it will be helpful to use the summation-by-parts formula

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{n} (a_0 + a_1 + \cdots + a_k)(b_k - b_{k+1}) \text{ where } b_{n+1} = 0.$$

Don't bother to give the easy induction proof of this formula but, for possible later use, remember it as the analog of integration by parts with sums replacing anti-derivatives and differences replacing derivatives.

5. #13, p.62. The sequence defined in this problem is an example of a continued fraction. A nice theory of continued fractions was developed in the first half of the twentieth century. Unfortunately, continued fractions are no longer popular and only a handful of people are aware of the nice theory.

6. #26, p. 64. Thus, x is a cluster point of the given set if there is an infinite sequence of distinct terms converging in $\mathbb{R}$ to x and each term in the sequence is the sum of the reciprocals of two natural numbers. This is a bit tricky. Start off by showing that $1/n$ is a cluster point for each $n \in \mathbb{N}$, 0 is also a cluster point, and all cluster points must be in [0,1]. Then try to determine whether or not there are other cluster points.

7. Problem on Norms for Function Spaces. (i) for $f \in \mathbb{C}_c^{\mathbb{N}}$ $= \{$functions from $\mathbb{N}$ into $\mathbb{C}$ for which there is some N_f $s.t.$ $f(k) = 0$

for all $k \geq \mathrm{N}_f\}$, we define $\|f||_\infty = max\{|f(k) : k \in \mathbb{N}\}$ and, for p $\in$ [1,∞), define $\|f||_p = \ = \{\sum_{k=1}^{\infty} |f(k)|^p\}^{1/p}$. Just as with the corresponding norms for $\mathbb{R}^n$, it's clear that $\|\cdot\|_p$ is a norm for p=1,2,∞ and is a Euclidean norm for p=2. For $p \neq$ 1,2,∞, it's obvious that $\|\cdot\|_p$ satisfies the positivity property N(1) and the scaling property N(2) but far from obvious that it satisfies the triangle inequality property N(3). In fact, N(3) is satisfied (we'll either prove this in class later in the semester or the proof will be posted on the course website) so $\|\cdot\|_p$ is a norm on $\mathbb{C}_c^{\mathbb{N}}$ for each real number $p \in$ [1,∞]. Show that no two of these norms are equivalent on $\mathbb{C}_c^{\mathbb{N}}$ although, for q>p, $\|f||_q \leq ||f||_p \, \forall f \in \mathbb{C}_c^{\mathbb{N}}$.

(ii) Let $\mathrm{C}_c(\mathbb{R}^n) = \{$continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ for which there is some $c_f \ > 0 \ s.t. \ f(x) = 0$ for each $x \in \mathbb{R}^n$ with $\|x||_\infty \geq c_f\}$. As we'll prove later, each $f \in \ \mathrm{C}_c(\mathbb{R}^n)$ is bounded with $|f|$ having maximum and minimum values and the Riemann integral $\int_{\mathbb{R}^n} f dx = \int_{\mathbb{R}^n} \mathrm{Re}(f) dx + i \int_{\mathbb{R}^n} \mathrm{Im}(f) dx$ is well defined. We define $||f||_\infty = max\{|f(x)| : x \in \mathbb{R}^n\}$ and, for $p \in [1, \infty)$, define $\|f||_p = \{\int_{\mathbb{R}^n} |f|^p dx\}^{1/p}$. The same tricks used to show that $\|\cdot\|_p$ is a norm on $\mathbb{C}_c^{\mathbb{N}}$ show that it is also a norm on $\mathrm{C}_c(\mathbb{R}^n)$. Accepting this, show that p, q in [1,∞] with $p \neq q$, $||\cdot||_p$ is not equivalent to $\|\cdot\|_q$. Just as in (i), this is shown by giving examples where $||f||_p$ is very large but $\|f||_q$ is small; it's enough to give such examples for $n = 1$ since 1-dimensional examples can easily be converted to $n-$ dimensional examples. You'll find it especially easy to give examples using functions whose graphs consist of two intersecting lines which, together with the $x - axis$, form a "tent".

$Remark: \ There \, are \, many \, applications \, for \, the \, function \, norms$ $|| \cdot \|_p, \, p \in [1, \infty] \, defined \, above \, and \, for \, their \, generalizations \, to$ $\mathbb{R} - valued \, or \, \mathbb{C} - \ valued \, functions \, on \, sets \, other \, than \, \mathbb{N} \, or \, \mathbb{R}^n.$ $Math \, 4121 \, looks \, at \, such \, norms \, when \, Riemann \, integrals \, are \, replaced$ $by \, Lebesgue \, integrals \, and \, continuous \, functions \, replaced \, by$ $''measurable \, functions'' \, which \, need \, not \, be \, continuous \, at \, any \, point.$