Fourier Transforms, Zak Transforms, and Plancherel Theorems for Abelian Groups

I. The real line case

1. Definition. (*i*) By $L^p(\mathbb{R})$, $1 \le p < \infty$, we mean the space of Lebesgue measurable functions $f : \mathbb{R} \mapsto \mathbb{C}$ for which $|f|^p$ is integrable and we denote $(\int_{\mathbb{R}} |f|^p \, dx)^{1/p}$ by $||f||_p$. Here dx is the increment of Lebesgue measure on $\mathbb{R}$. By routine measure theory, $(L^p(\mathbb{R}), || \, . \, ||_p)$ is a separable Banach space (separable meaning that it has countable dense subsets) in which the space $C_c(\mathbb{R})$ of compactly supported continuous functions is dense.

(*ii*) The Fourier transform $\widehat{f}$ of $f \in L^1(\mathbb{R})$ is the function defined on a copy $\widehat{\mathbb{R}}$ of $\mathbb{R}$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx. \qquad (1)$$

Using the density of $C_c(\mathbb{R})$ in $L^1(\mathbb{R})$, it follows that $\widehat{f}$ is a bounded, continuous function on $\widehat{\mathbb{R}}$.

2. The Plancherel Theorem for $\mathbb{R}$ assets that there is a unitary operator (1-1 onto linear isometry) $\mathcal{F}: L^2(\mathbb{R}) \mapsto L^2(\widehat{\mathbb{R}})$ for which $\mathcal{F}f = \widehat{f}$ when $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and, when $g \in L^1(\widehat{\mathbb{R}}) \cap L^2(\widehat{\mathbb{R}})$,

$$(\mathcal{F}^{-1} g)(x) = \widehat{g}(-x) = \int_{\widehat{\mathbb{R}}} g(\xi) e^{2\pi i \xi x} d\xi \qquad (2)$$

Textbooks on real analysis typically prove this theorem by identifying $\widehat{\mathbb{R}}$ with $\mathbb{R}$, constructing a dense subspace $\mathcal{S}_1$ of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ mapped isometrically by $f \mapsto \widehat{f}$ onto a dense subspace $\mathcal{S}_2$ of $L^2(\mathbb{R})$, and using the easy general result that an isometry from a dense subspace of a Hilbert space $\mathcal{H}_1$ onto a dense subspace of another Hilbert space $\mathcal{H}_2$ has a unique extension to a unitary map from $\mathcal{H}_1$ onto $\mathcal{H}_2$. The three most common methods to do this are as follows.

Method I. Use complex variables to deduce that the Gaussion function $g_o(x) = e^{-\pi x^2}$ is its own Fourier transform, then go through a long series of algebraic computations to show that $f \mapsto \widehat{f}$ is an isometry on the space $\mathcal{S}$ consisting of polynomials times g_0 and that $\mathcal{S}$ is dense in $L^2(\mathbb{R})$.

Method II. Using the fact that g_o is its own Fourier transform, go through lengthy approximate identity arguments ultimately leading to showing that $f \mapsto \widehat{f}$ is an isometry from $C_c(\mathbb{R})$ onto a dense subspace of $L^2(\mathbb{R})$. .

Method III (the 19th century method). By rescaling, one can describe compactly supported continuous functions by converging Fourier series, then approximate the Fourier integral (1) by Riemann sums, and give painful arguments showing that, in the limit as the step size in the Riemann sums goes to 0, the isometry property for $\mathcal{F}$ and the inverse formula (2) "emerge".

Each of Methods I-III is non-elementary and full details are lengthy and tedious. Only Method III relates Fourier series to Fourier transforms but does so in a very clumsy way. While each of these 3 methods extends in a routine way to obtain the analogous Plancherel theorem for $\mathbb{R}^n$, none of these methods gives an insight into why there "ought" to be a Plancherel Theorem for every "reasonable" abelian group and how one might go about proving such a general theorem.

3. **History of the "Zak transform proof" of the Plancherel Theorem.** In his 1940 book on integration on topological groups, Andre Weil outlined an "elementary" proof of the Plancherel Theorem for $\mathbb{R}$ and other abelian topological groups based entirely on Fourier series but much more direct and easier to understand than the ugly Method III proof. Weil also alluded to

this in his 1964 paper on integral operators for locally compact abelian groups. In between, I. Gelfand mentioned Weil's elegant 1940 proof in a 1950 paper on eigenfunction expansions. In 1967, an Israeli physicist named J. Zak properly attributed to Weil a certain "Fourier series" transform operator converting $L^2(\mathbb{R})$ to $L^2([0,1]^2)$ and discussed its properties without mentioning Weil's use of this operator to obtain the Plancherel Theorem. Engineers and others started referring to this operator as the Zak transform and began using it heavily for other purposes. Karl-Heinz Groechenig discussed some of the connections between the Fourier transform and the Zak transform in his 2001 book on Time Frequency Analysis. Professor Weiss and I rediscovered Weil's proof several years ago, were certain that "someone" must have discovered it earlier (we learned about Weil's observations only recently), and remain astounded that, with only a few exceptions (notably, Groechenig and G. Folland), experts in real analysis are unaware of it and continue to teach courses and write books giving the ponderous Method I-Method III proofs that have turned generations of students against anything having to do with Fourier transforms.

4. The Zak Transform Proof of the Plancherel Theorem.

(i) The Zak transform Zf of $f \in C_c(\mathbb{R})$ is the function of two real variables defined by

$$Zf(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i\xi k} \qquad (3)$$

Obviously, the right hand side of (3) is a finite Fourier series in ξ for each x so Φ=Zf is jointly continuous in (x,ξ), $1-$periodic in ξ, and by a simple change of summation index computation, satisfies the transformation condition

$$\Phi(x+j,\xi) = e^{2\pi i\xi j}\,\Phi(x,\xi) \qquad (4)$$

which means $|\Phi(x,\xi)|$ is $1-$periodic in x as well as in ξ. With $e_k(\xi) = e^{2\pi i\xi k}$, the integral of e_k over any interval of length 1 is 0 if $k \neq 0$ and 1 if $k = 0$. This gives the simple inversion formula

$$f(x) = (\mathrm{Z}^{-1}\Phi)(x) = \int_0^1 \Phi(x,\xi)d\xi \qquad (5)$$

Also, since $\xi \mapsto \mathrm{Z}f(x,\xi)$ is the Fourier series with k^{th} coefficient $f(x+k)$, the Plancherel Theorem for the group $\mathbb{T}$=$\mathbb{R}/\mathbb{Z}$ and dual group $\mathbb{Z}$ yields

$$\int_0^1 \{\sum_{k\in\mathbb{Z}} |f(x+k)|^2\}dx = \int_0^1\int_0^1 |\mathrm{Z}f(x,\xi)|^2 d\xi\, dx \qquad (6)$$

Interchanging $\int_0^1$ and $\sum\limits_{k\in\mathbb{Z}}$ on the left in (6) and then using the substitution $t = x + k$ for each k yields $\sum\limits_{k\in\mathbb{Z}} \int_k^{k+1} |f(t)|^2 dt = ||f||_2^2$ so (6) becomes the statement that $f \mapsto \mathrm{Z}f$ is an isometry provided $||\mathrm{Z}f||$ is understood to be the L^2 norm of the doubly 1-periodic function $|\mathrm{Z}f|$.

(ii) By the extension principle mentioned above, $f \mapsto \mathrm{Z}f$ extends uniquely to a unitary map still called Z from the Hilbert space $\mathrm{L}^2(\mathbb{R})$ onto the Hilbert space $\mathcal{H}$ of measurable functions $\Phi(x,\xi)$ which are 1-periodic in ξ, satisfy the transformation condition (4) in x, and for which

$$||\Phi||^2 = \int\int_{[0,1]^2} |\Phi(x,\xi)|^2\, dx\, d\xi < \infty. \qquad (7)$$

Moreover, (5) continues to define the inverse of Z.

(iii) Reversing the roles of x and ξ as well as changing the sign of exponents, we have another Zak transform $\mathrm{Z}^\sim$ defined on $\mathrm{L}^2(\widehat{\mathbb{R}})$ by the Fourier series

$$\mathrm{Z}^\sim g(x,\xi) = \sum_{j\in\mathbb{Z}} g(\xi+j)e^{2\pi ijx}. \qquad (8)$$

in x with j^{th}Fourier coefficient $g(\xi+j)$.

As in *(ii)*, $\mathrm{Z}^\sim$ is an isometry from $\mathrm{L}^2(\widehat{\mathbb{R}})$ onto the Hilbert space $\mathcal{H}^\sim$ consisting of measurable functions $\Phi^\sim(x,\xi)$ which are 1-periodic in x, satisfy the transformation condition

$$\Phi^\sim(x,\xi+l) = e^{-2\pi ilx}\Phi^\sim(x,\xi), \qquad (9)$$

in ξ, and $||\Phi^\sim||^2 = \int\int_{[0,1]^2} |\Phi^\sim(x,\xi)|^2\, dx\, d\xi < \infty. \qquad (10)$

Also, the inverse of $Z^{\sim}$ is the analog of (5), namely

$$((Z^{\sim})^{-1}\Phi^{\sim})(\xi) = \int_{[0,1]}\Phi^{\sim}(x,\xi)dx \tag{11}$$

(iv) By a simple computation, we have a unitary map $\mathcal{U}{:}\mathcal{H}\to\mathcal{H}^{\sim}$ defined by

$$\Phi^{\sim}(x,\xi) = (\mathcal{U}\Phi)(x,\xi) = e^{-2\pi i\xi x}\Phi(x,\xi). \tag{12}$$

This leads to defining $\mathcal{F}$ to be the composition $(Z^{\sim})^{-1}\circ\mathcal{U}\circ Z$ of the unitary map Z from $L^2(\mathbb{R})$ onto $\mathcal{H}$, the unitary map $\mathcal{U}$ from $\mathcal{H}$ onto $\mathcal{H}^{\sim}$, and the unitary map $(Z^{\sim})^{-1}$ from $\mathcal{H}^{\sim}$ onto $L^2(\widehat{\mathbb{R}})$. Thus, $\mathcal{F}{:}L^2(\mathbb{R})\to L^2(\widehat{\mathbb{R}})$ is unitary. For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, use of the Fubini theorem to interchange $\int$ and $\sum$ plus the change of variable $t = x + k$ yields

$$\begin{aligned}(\mathcal{F}f)(\xi) &= \int_{[0,1]} e^{-2\pi i\xi x}\sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i\xi k}\, dx\\ &= \sum_{k\in\mathbb{Z}}\int_{[0,1]} f(x+k)e^{-2\pi i\xi(x+k)}dx\\ &= \sum_{k\in\mathbb{Z}}\int_{[k,k+1]} f(t)e^{-2\pi i\xi t}dt\\ &= \int_{\mathbb{R}} f(t)e^{-2\pi i\xi t}dt\\ &= \widehat{f}(\xi)\end{aligned} \tag{13}$$

for all ξ. By an analagous computation using $\mathcal{F}^{-1} = Z\circ\mathcal{U}^{-1}\circ Z^{\sim}$, we obtain (2). This completes the proof of the theorem.

$Remarks$: (a) One of the amusing consequences of the identify $Z^{\sim}\widehat{f} = \mathcal{U}Zf$ is the famous Poisson summation formula

$$\sum_{k\in\mathbb{Z}} |f(k)|^2 = \sum_{j\in\mathbb{Z}} |\widehat{f}(j)|^2 \tag{14}$$

obtained by evaluating the identity at $(x, \xi) = (0, 0)$ when f satisfies smoothness and decay conditions sufficient to make both Zf and $Z^{\sim}\widehat{f}$ continuous at (0,0); not surprisingly, classical proofs of the Poisson summation formula are much harder to understand.

(b) The heart of the above proof of the Plancherel Theorem for $\mathbb{R}$ is the easy manipulations in (13) plus the decomposition of $\mathcal{F}$ as the composition of 3 unitary maps. The only "hard" result used in the proof is the Plancherel Theorem for Fourier series (proved in class using the Poisson kernel for $\mathbb{D}$). Since Zak transforms are just special kinds of Fourier series, one can summarize the proof by saying that everything about Fourier transforms reduces to Fourier series properties.

II. LCA groups and their duals.

(i) A locally compact abelian group (LCA group) is an additive group (G,+) together with a topology on G which is separable, locally compact, and Hausdorff such that the map $(x, y) \mapsto x - y$ is continuous from G $\times$ G into G. The building blocks for all LCA groups are the groups $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$=$\mathbb{R}/\mathbb{Z}$, and, for $n \geq 2$, the finite cyclic group $\mathbb{Z}_n$=$\mathbb{Z}/n\mathbb{Z}$ with n elements. The groups which are isomorphic and homeomorphic to a finite product of building block groups are precisely the compactly generated abelian groups.

(ii) For a general LCA group G, Weil showed that the connected component G_0 of 0 in G is isomorphic and homeomorphic to the direct product of $\mathbb{R}^n$ (for some non-negative integer n) and a closed subgroup of the Tychonoff product of countably many copies of $\mathbb{T}$; *vice versa*, all topological abelian groups with this property are LCA groups.

(iii) A character of a LCA group G is a continuous homomorphism from G into the multiplicative group $\partial\mathbb{D}$. It's

convenient to use the isomorphism and homeomorphism $\theta+\mathbb{Z} \mapsto e^{2\pi i\theta}$ from $\mathbb{T}$ onto $\partial\mathbb{D}$ to parametrize the multiplicative group of characters of G by members of an additive group $\widehat{\mathrm{G}}$ called the dual group of G. Then $(\xi,x) \mapsto \xi \cdot x$ is a bi-additive map from $\widehat{\mathrm{G}} \times \mathrm{G}$ into $\mathbb{T}$ and a typical character of G has the form $x \mapsto e_\xi(x) = e^{2\pi i\xi\cdot x}$ for a unique $\xi \in \widehat{\mathrm{G}}$. Using the fact that G is locally compact, one can show that the socalled compact-open topology on $\widehat{\mathrm{G}}$ turns $\widehat{\mathrm{G}}$ into a LCA group for which $(\xi, x) \mapsto \xi \cdot x$ is continuous and every character of $\widehat{\mathrm{G}}$ has the form $\xi \mapsto e_x(\xi)$ for a unique $x \in \mathrm{G}$. In this sense, G is the dual of $\widehat{\mathrm{G}}$.
[As a special case, when G is a compactly generated abelian Lie group, so is $\widehat{\mathrm{G}}$; indeed, for some non-negative integers n, m, l, and some finite group F, $\mathrm{G} \cong \mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^l \times \mathrm{F}$ while $\widehat{\mathrm{G}} \cong \mathbb{R}^n \times \mathbb{T}^l \times \mathbb{Z}^m \times \mathrm{F}$ so $\mathrm{G} \cong \widehat{\mathrm{G}} \Leftrightarrow m = l$].

(iv) For the special case of a compact abelian group K, $\widehat{\mathrm{K}}$ is a countable discrete group and there is a unique translation-invariant Borel measure ν on K for which $\nu(\mathrm{K})=1$;it follows easily that the characters of K are mutually perpendicular unit vectors in $\mathrm{L}^2(\mathrm{K},\nu)$. Weil's description of K leads to a description of $\widehat{\mathrm{K}}$ as a quotient of the additive sum of countably many copies of $\mathbb{Z}$. Either using this or, more simply, using the Stone-Weierstrass theorem, it follows that the algebra generated by the characters of K is dense in the space of continuous functions on K and hence is dense in $\mathrm{L}^2(\mathrm{K},\nu)$. This leads to the Plancherel theorem for K (or generalized Fourier series theorem) stating that every $f \in \mathrm{L}^2(\mathrm{K},\nu)$ has the Fourier series description $\sum_{j\in\widehat{\mathrm{K}}} < f, e_j > e_j$ with $f \mapsto (<f,e_j>)_{j\in\widehat{\mathrm{K}}}$ a unitary map from $\mathrm{L}^2(\mathrm{K},\nu)$ onto $l^2(\widehat{\mathrm{K}})$ and $\{e_j : j \in \widehat{\mathrm{K}}\}$ an orthonormal basis for $\mathrm{L}^2(\mathrm{K},\nu)$.

III. Outline of the Zak transform proof of the Plancherel Theorem for a general non-compact LCA group G.

(i) G contains many lattices, $i.e.$ topologically discrete subgroups $\mathcal{L} \subset$ G for which K=G/$\mathcal{L}$ is compact in the quotient topology. For each choice of a lattice $\mathcal{L} \subset$ G, $\mathcal{L}^{\perp} = \{j \in \widehat{G} : j \cdot k = 0 + \mathbb{Z}\, \forall k \in \mathcal{L}\}$ is a lattice in $\widehat{G}$ called the lattice dual to $\mathcal{L}$ and $\mathcal{L}^{\perp}$ can be identified with $\widehat{K}$. Also, $\mathcal{L}$ determines a unique translation invariant Borel measure λ on G for which λ(C) = 1 for every Borel subset C $\subset$ G such that G is the disjoint union of the sets C+k, $k \in \mathcal{L}$; any such C is said to be a $\mathcal{L}$-tiling set. Similarly, $\mathcal{L}^{\perp}$ uniquely determines a translation invariant Borel measure $\lambda^{\perp}$on $\widehat{G}$ assigning the measure 1 to every $\mathcal{L}^{\perp}$ tiling set $C^{\perp} \subset \widehat{G}$.

(ii) Associated with each lattice pair $\mathcal{L}$ and $\mathcal{L}^{\perp}$ as in (ii) are Zak transforms Z=$Z_{\mathcal{L}}$ and $Z^{\sim} = Z^{\sim}_{\mathcal{L}^{\perp}}$defined for $f \in L^2(G,\lambda)$ and $g \in L^2(\widehat{G}, \lambda^{\perp})$ by the analogs of (1) and (8), namely

$$Zf(x,\xi) = \sum_{k\in\mathcal{L}} f(x+k)e_{-k}(\xi) \tag{15}$$

$$Z^{\sim}g(x,\xi) = \sum_{j\in\mathcal{L}^{\perp}} g(\xi+j)e_j(x). \tag{16}$$

Just as in the real line case, the theory of Fourier series mentioned in **II** is all that's needed to check that Z and $Z^{\sim}$ are unitary maps onto Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^{\sim}$ of functions satisfying transformation conditions analagous to those in (4) and (9) and having magnitudes which are square integrable on the compact group G $\times$ $\widehat{G}/\mathcal{L} \times \mathcal{L}^{\perp}$. Moreover, $(\mathcal{U}\Phi)(x,\xi) = e^{-2\pi i\xi\cdot x}\Phi(x,\xi)$ continues to define a unitary map from $\mathcal{H}$ onto $\mathcal{H}^{\perp}$. Defining $\mathcal{F}$=$(Z^{\sim})^{-1}\circ\mathcal{U}\circ$Z, $\mathcal{F}$ is automatically unitary from $L^2(G,\lambda)$ onto $L^2(\widehat{G}, \lambda^{\perp})$ and the same computation as in (13) shows that, for $f \in L^2(G,\lambda) \cap L^1(G,\lambda)$ and $g \in L^2(\widehat{G},\lambda^{\perp}) \cap L^1(\widehat{G},\lambda^{\perp})$

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_G f(x)e^{-2\pi i\xi\cdot x}\,d\lambda(x) \tag{17}$$

$$(\mathcal{F}^{-1}g)(x) = \widehat{g}(-x) = \int_{\widehat{G}} g(\xi)e^{2\pi i\xi\cdot x}\,d\lambda^{\perp}(\xi) \tag{18}$$

with the statements (17) and (18) being the Plancherel Theorem for G. Basically, accepting the standard structural results in (i) along with the Fourier series result stated in **II** (iv), this general proof is "the same" as in the real line case and serves to underline the key point that everything about Fourier transforms is an easy corollary of Fourier series results. One can go on to prove many more results about Zak transforms and Fourier transforms and use them for a host of practical applications.