Geometry in (Very) High Dimensions

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The RGB color cube

Blue = (0,0,1)
Cyan = (0,1,1)
Magenta = (1,0,1)
White = (1,1,1)
Black = (0,0,0)
Green = (0,1,0)
Red = (1,0,0)
Yellow = (1,1,0)
3-dimensional space of three-note chords, with major and minor triads found near the center. (An orbifold, much like a 3-dim Moebius band.) Tymoczko is composer and professor of music at Princeton University.
The 6 basic emotions
Based on research by Paul Ekman

Other facial expressions of emotions are combinations of these 6, just as colors are combinations of R, B, G (a 6-dimensional space.)

Illustration by Scott McCloud
From *Making Comics* by McCloud (2006)
Combinations of primary faces

- Anger + Disgust = Outrage
- Fear + Joy = Desperation
- Anger + Fear = Caged Animal
- Anger + Joy = Cruelty
- Anger + Sadness = Betrayal
- Anger + Surprise = "What the --?!"
- Fear + Sadness = Devastation
- Fear + Surprise = Spooked
- Joy + Sadness = Faint Hope
- Joy + Surprise = Amazement
Thermodynamic space

To specify the kinetic state of one mole of gas we need:

3 velocity components for each of $6.022 \times 10^{23}$ molecules.

So we need more than $18 \times 10^{23}$ dimensions.
Space $\mathbb{R}^n$ consists of all $n$-tuples of real numbers: $x = (x_1, \ldots, x_n)$.

Distance between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$:

$$d(x, y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}$$

Euclid (c. 300 BC)  
Descartes (17th century)
Cubes and Balls in Cartesian $n$-space

The $n$-cube $C(r)$ of side length $2r$: set of $x$ such that $-r \leq x_i \leq r.$

$C^1(r) = [-r, r]$

$C^2(r) = [-r, r]^2$

$C^3(r) = [-r, r]^3$

It is also the Cartesian product: $C^n(r) = [-r, r] \times \cdots \times [-r, r]$ (n times.)
$n$-dimensional balls

The pythagorean formula in dimension $n$ is

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2} = \text{distance from } x \text{ to origin } (0, \ldots, 0).$$

The $n$-ball of radius $r$ is the set of $x$ at distance $\leq r$ from the origin:

$$\mathcal{B}^n(r) = \{ x \in \mathbb{R}^n : |x| \leq r \}. $$

$\mathcal{B}^1(r)$ $\mathcal{B}^2(r)$ $\mathcal{B}^3(r)$
Testing our geometric intuition ...
Similar arrangement in dimension 3
What happens to the red ball as $n$ goes to infinity?

Now, do the same in arbitrary dimensions:

- Begin with a cube of dimension $n$ and side length 1.
- Pack $2^n$ balls inside, each of radius $\frac{1}{4}$.
- Add a (red) ball at the center that touches all the other balls.
- Let $r_n$ be the radius of the red ball.

What happens to $r_n$ when $n$ goes to infinity?
\( r_n \rightarrow \infty \)

In dimension 2,

\[
\begin{align*}
    r_2 &= \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} - \frac{1}{4} = \frac{1}{4} (\sqrt{2} - 1). \\

\end{align*}
\]

In dimension \( n \),

\[
\begin{align*}
    r_n &= \sqrt{\left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^2} - \frac{1}{4} = \frac{1}{4} (\sqrt{n} - 1) \\

\end{align*}
\]
The fundamental volume is that of the cube. We define it to be

\[ \text{Vol}(\mathcal{C}^n(a/2)) = \text{Vol}(n\text{-cube of side length } a) = a^n \]

By filling the ball with small cubes and taking a limit,

\[ \text{Vol}(\mathcal{B}^n(r)) = \text{Vol}(\mathcal{B}^n(1)) r^n. \]

Using multivariate calculus and induction in the dimension,

\[ \text{Vol}(\mathcal{B}^n(1)) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \sim \frac{1}{\sqrt{n\pi}} \left(\frac{\sqrt{2\pi e}}{n}\right)^n. \]

This is very small for big \( n \).
Example: dimension 100

- Volume of cube = 1
- Volume of inscribed ball = $2.37 \times 10^{-40}$
How are volumes distributed? (1. Shells)

Fraction of volume of \( n \)-ball contained in the outer shell of thickness \( a \):

\[
\frac{\text{Vol}(\mathbb{B}^n(r)) - \text{Vol}(\mathbb{B}^n(r-a))}{\text{Vol}(\mathbb{B}^n(r))} = \frac{r^n - (r-a)^n}{r^n} = 1 - \left(1 - \frac{a}{r}\right)^n.
\]

Therefore, for arbitrarily small \( a \), as \( n \to \infty \),

\[
\frac{\text{Vol}(\text{shell of thickness } a)}{\text{Vol}(\mathbb{B}^n(r))} \to 1.
\]

The volume inside a ball (for big \( n \)) is mostly near the surface.
How are volumes distributed? (2. Central slabs)

Consider the $n$-dimensional ball of fixed volume equal to 1.

It can be shown that:

- radius $r_n \sim \sqrt{\frac{n}{2\pi e}}$

- about 96% of the volume lies in the (relatively very thin) slab

$$\left\{ x \in \mathbb{B}^n(r_n) : -\frac{1}{2} \leq x_1 \leq \frac{1}{2} \right\}.$$

This is true for the slab centered on any $(n-1)$-dimensional subspace!
Hyper-area of slices

Associated to the \( n \)-dimensional cube of side 1, define:

- \( e = \frac{1}{\sqrt{n}}(1, \ldots, 1) \), a vector of unit length;
- \( V_{n-1}(t) = \) hyper-area of slice perpendicular to \( e \) above \( t \).

Theorem (CLT)

\[
V_{n-1}(t) = \text{Vol}_{n-1} \left( \left\{ x : \left| \frac{x_1 + \cdots + x_n}{\sqrt{n}} \right| = t \right\} \right) \sim \sqrt{\frac{6}{\pi}} e^{-6t^2}
\]
If we identify Probability $\Leftrightarrow$ Volume ...

Theorem (Central Limit Theorem)

*If* $X_1, X_2, \ldots$ are random numbers (uniformly distributed) on $\left[ -\frac{1}{2}, \frac{1}{2} \right]$, 

$$
\lim_{n \to \infty} \Pr \left( \left| \frac{X_1 + \cdots + X_n}{\sqrt{n}} \right| \leq x \right) = \int_{-x}^{x} \sqrt{\frac{6}{\pi}} e^{-6t^2} \, dt
$$

Probability experiment:

Compute 5000 sample values of the random number

$$
Y = \frac{X_1 + \cdots + X_{10}}{\sqrt{10}}
$$

then plot the distribution of values of $Y$ (histogram)
Morals to draw from the story

- Concepts in dimension 2, 3 often generalize to $n > 3$;
- But peculiar things can happen for big $n$.
- Insight is regained by thinking probabilistically.
- Probability theory can be “interpreted” geometrically.

Algebra, analysis, combinatorics, modern physical theories, music, and many other subjects use geometric ideas to represent concepts that may not seem geometric at first.

This is reflected in the many flavors of modern geometry:
- Riemannian and pseudo-Riemannian geometries;
- algebraic geometry;
- symplectic and non-commutative geometry;
- information geometry, etc., etc., etc.