Homework set 8 - due 11/13/26

Math 3010 – Renato Feres

Note 1. Work on all the problems listed below, but only those marked with an $(*)$ need to be turned in.
Note 2. The problems marked with $(*)$ are all worth the same number of points.

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1. **Read Chapters 12 and 13 of Liebeck's textbook.**

2. **Chapter 12, Exercise 3** $(*)$ For a positive integer n, define $\phi(n)$ to be the number of positive integers $a < n$ such that $\text{hcf}(a,n) = 1$. (For example, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$.)

   (a) Work out $\phi(n)$ for $n = 5, 6, \ldots, 10$.

   (b) If p is a prime, show that $\phi(p) = p - 1$ and, more generally, that $\phi(p^r) = p^r - p^{r-1}$.

3. **Chapter 13, Exercise 2** $(*)$ Let p be a prime number and k a positive integer.

   (a) Show that if x is an integer such that $x^2 \equiv x \bmod p$, then $x \equiv 0$ or $1 \bmod p$.

   (b) Show that if x is an integer such that $x^2 \equiv x \bmod p^k$, then $x \equiv 0$ or $1 \bmod p^k$.

4. **Chapter 13, Exercise 3** $(*)$ For each of the following congruence equations, either find a solution $x \in \mathbb{Z}$ or show that no solution exists:

   (a) $99x \equiv 18 \bmod 30$

   (b) $91x \equiv 84 \bmod 143$

   (c) $x^2 \equiv 2 \bmod 5$

   (d) $x^2 + x + 1 \equiv 0 \bmod 5$

   (e) $x^2 + x + 1 \equiv 0 \bmod 7$

5. **Chapter 13, Exercise 6** $(*)$ Let p be a prime number, and let a be an integer that is not divisible by p. Prove that the congruence equation $ax \equiv 1 \bmod p$ has a solution $x \in \mathbb{Z}$.

6. **Chapter 13, Exercise 9.** Let p be a prime and let $\overline{a}, \overline{b} \in \mathbb{Z}_p$, with $\overline{a} \neq 0$. Prove that the equation $\overline{ax} = \overline{b}$ has a solution for $\overline{x} \in \mathbb{Z}_p$.

7. **Chapter 13, Exercise 10** $(*)$

   (a) Construct the addition and multiplication tables for $\mathbb{Z}_6$.

   (b) Find all solutions in $\mathbb{Z}_6$ of the equation $\overline{x}^2 + \overline{x} = 0$.