Homework set 8 - due 11/15/19
Math 310 – Renato Feres

Note 1. Work on all the problems listed below, but only those marked with an (∗) need to be turned in.
Note 2. The problems marked with (∗) are all worth the same number of points.
Note 3. Chapters and exercise numbers refer to the 4th edition of Liebeck’s text.

1. Read Chapters 12 and 13 of Liebeck’s textbook.

2. Chapter 12, Exercise 3 (∗) For a positive integer $n$, define $\phi(n)$ to be the number of positive integers $a < n$ such that $\text{hcf}(a, n) = 1$. (For example, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$.)
   (a) Work out $\phi(n)$ for $n = 5, 6, \ldots, 10$.
   (b) If $p$ is a prime, show that $\phi(p) = p - 1$ and, more generally, that $\phi(p^r) = p^r - p^{r-1}$.

3. Chapter 13, Exercise 2 (∗) Let $p$ be a prime number and $k$ a positive integer.
   (a) Show that if $x$ is an integer such that $x^2 \equiv x \mod p$, then $x \equiv 0$ or $1 \mod p$.
   (b) Show that if $x$ is an integer such that $x^2 \equiv x \mod p^k$, then $x \equiv 0$ or $1 \mod p^k$.

4. Chapter 13, Exercise 3 (∗) For each of the following congruence equations, either find a solution $x \in \mathbb{Z}$ or show that no solution exists:
   (a) $99x \equiv 18 \mod 30$
   (b) $91x \equiv 84 \mod 143$
   (c) $x^2 \equiv 2 \mod 5$
   (d) $x^2 + x + 1 \equiv 0 \mod 5$
   (e) $x^2 + x + 1 \equiv 0 \mod 7$

5. Chapter 13, Exercise 6 (∗) Let $p$ be a prime number, and let $a$ be an integer that is not divisible by $p$. Prove that the congruence equation $ax \equiv 1 \mod p$ has a solution $x \in \mathbb{Z}$.

6. Chapter 13, Exercise 9. Let $p$ be a prime and let $\overline{a}, \overline{b} \in \mathbb{Z}_p$, with $\overline{a} \neq 0$. Prove that the equation $\overline{a}x = \overline{b}$ has a solution for $x \in \mathbb{Z}_p$.

7. Chapter 13, Exercise 10 (∗)
   (a) Construct the addition and multiplication tables for $\mathbb{Z}_6$.
   (b) Find all solutions in $\mathbb{Z}_6$ of the equation $\overline{x}^2 + \overline{x} = 0$. 