Note 1. Work on all the problems listed below, but only those marked with an (\*) need to be turned in.

Note 2. The problems marked with (\*) are all worth the same number of points.

Note 3. Chapters and exercise numbers refer to the 4th edition of Liebeck's text.

1. Read Chapter 16 of Liebeck's textbook.

2. Chapter 16, Exercise 12 (\*) The digits 1,2,3,4,5,6 are written down in some order to form a six-digit number.
   (a) How many such six-digit numbers are there altogether?
   (b) How many such numbers are even?
   (c) How many are divisible by 4?
   (d) How many are divisible by 8? (Hint: First show that the remainder on dividing a six-digit number $abcdef$ by 8 is $4d + 2e + f$.)

   (a) Find the coefficient of $x^{15}$ in $(1 + x)^{18}$.
   (b) Find the coefficient of $x^4$ in $\left(2x^3 - \frac{1}{x^2}\right)^8$.
   (c) Find the constant term in the expansion of $\left(y + x^2 - \frac{1}{xy}\right)^{10}$.

4. Read Chapter 17 of Liebeck's textbook.

5. Chapter 17, Exercise 2 (\*) Which of the following statements are true and which are false? Give proofs or counterexamples.
   (a) For any sets $A, B, C$, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
   (b) For any sets $A, B, C$, we have $(A - B) - C = A - (B - C)$.
   (c) For any sets $A, B, C$, we have $(A - B) \cup (B - C) \cup (C - A) = A \cup B \cup C$.

6. Chapter 17, Exercise 3. Work out $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$, where $A_n$ is defined as follows for $n \in \mathbb{N}$:
   (a) $A_n = \{x \in \mathbb{R} : x > n\}$
7. Chapter 17, Exercise 4 (*)

(a) \(73\% \text{ of British people like cheese, } 76\% \text{ like apples and } 10\% \text{ like neither. What percentage like both cheese and apples?}\)

(b) \(\text{In a class of 30 children, everyone supports at least one of three teams: 16 support Manchester United, 17 support Stoke City and 14 support Doncaster Rovers; also 8 support both United and City, 7 both United and Rovers, and 9 both City and Rovers. How many support all three teams?}\)

8. Read Chapter 19 of Liebeck's textbook.

9. Chapter 19, Exercise 4 (*) Let \(X, Y, Z\) be sets and let \(f : X \to Y\) and \(g : Y \to Z\) be functions. In each of the following, give a proof or a counterexample.

(a) Given that \(g \circ f\) is onto, can you deduce that \(f\) is onto?

(b) Given that \(g \circ f\) is onto, can you deduce that \(g\) is onto?

(c) Given that \(g \circ f\) is 1-1, can you deduce that \(f\) is 1-1?

(d) Given that \(g \circ f\) is 1-1, can you deduce that \(g\) is 1-1?

10. Chapter 19, Exercise 5. The Pigeonhole Principle (page 165 of Liebeck's text) states that: If we put \(n + 1\) or more pigeons into \(n\) pigeonholes, then there must be a pigeonhole containing more than one pigeon. Use this principle to prove the following statements involving a positive integer \(n\):

(a) In any set of 6 integers, there must be two whose difference is divisible by 5.

(b) In any set of \(n + 1\) integers, there must be two whose difference is divisible by \(n\).

(c) Given any \(n\) integers \(a_1, a_2, \ldots, a_n\), there is a non-empty subset of these whose sum is divisible by \(n\). (Hint: Consider the integers \(0, a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_n\) and use (b).)

(d) Given any set \(S\) consisting of ten distinct integers between 1 and 50, there are two different 5-element subsets of \(S\) with the same sum.

(e) Given any set \(T\) consisting of nine distinct integers between 1 and 50, there are two disjoint subsets of \(T\) with the same sum.

(f) In any set of 101 integers chosen from the set \(\{1, 2, \ldots, 200\}\), there must be two integers such that one divides the other.

11. Read Chapter 20 of Liebeck's textbook.

12. Chapter 20, Exercise 2. Let \(f\) and \(g\) be the following permutations in \(S_7\):

\[
f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 7 & 2 & 6 & 4 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 7 & 6 & 4 & 5 & 2 \end{pmatrix}.
\]

(a) Write down in cycle notation the permutations

\(f, g, g^2, g^3, f \circ g, (f \circ g)^{-1}\) and \(g^{-1} \circ f^{-1}\).
(b) What is the order of \( f \)? What is the order of \( g \)?

13. **Chapter 20, Exercise 4** (*) A pack of \( 2n \) cards is shuffled by the “interlacing” method described in Example 20.7 — in other words, if the original order is 1, 2, 3, \ldots, \( 2n \), the new order after the shuffle is 1, \( n + 1 \), 2, \( n + 2 \), \ldots, \( n \), \( 2n \). Work out how many times this shuffle must be repeated before the cards are again in the original order in the following cases [Note: doing only parts (a) and (b) is enough.]:

   (a) \( n = 10 \)
   (b) \( n = 12 \)
   (c) \( n = 16 \)
   (d) \( n = 24 \)
   (e) \( n = 26 \) (i.e., a real pack of cards).

The following problems are extra. You won’t see them in our final exam, but they have to do with too fundamental a topic to be completely left out from this course. You’ll need to know these concepts (countable and uncountable sets, Cantor’s diagonal argument, \( \mathbb{Q} \) is countable, \( \mathbb{R} \) is uncountable) in your future math courses.

**Extra 1. Read Chapter 21 of Liebeck’s textbook.**

**Extra 2. Chapter 21, Exercise 1.**

   (a) Show that if \( A \) is a countable set and \( B \) is a finite set, then \( A \cup B \) is countable.
   (b) Show that if \( A \) and \( B \) are both countable sets, then \( A \cup B \) is countable.

**Extra 3. Chapter 21, Exercise 2.**

   (a) Show that if each of the sets \( S_n \) (\( n = 1, 2, 3, \ldots \)) is countable, then the union \( S = \bigcup_{n=1}^{\infty} S_n \) is also countable.
   (b) Show that if \( S \) and \( T \) are countable sets, the the Cartesian product \( S \times T \) is also countable. Hence show that \( \bigcup_{n=1}^{\infty} S^n \) is countable, where \( S^n = S \times S \times \cdots \times S \) (\( n \) times).

**Extra 4. Chapter 21, Exercise 4.** Let \( S \) be the set consisting of all infinite sequences of 0s and 1s (so a typical member of \( S \) is \( 010011011100110 \ldots \), going on forever.) Use Cantor’s diagonal argument to prove that \( S \) is uncountable.