1. **(Exercise 9.4; 28 points.)** Consider a convex polyhedron whose surface consists of 9 squares and \(m\) octagons. Let \(V, E, F\) be, respectively, the number of vertices, edges, and faces of the polyhedron.

(a) Give a succinct definition of a *polyhedron*.

(b) What does it mean for the polyhedron to be *convex*?

(c) What is the value of \(F\)? (Express it in terms of \(m\). No need to justify.)

(d) What is the value of \(E\)? (Express it in terms of \(m\). No need to justify.)

(e) State Euler’s formula in general. (Don’t forget to state the conditions under which it is true.)

(f) What is the value of \(V\)? (Express it in terms of \(m\).)

(g) Taking for granted \(2F \geq V + 4\) (proved in a homework assignment) find the maximum possible value of \(m\).

**Solution.**

(a) A *polyhedron* is a solid whose surface consists of polygons (faces) such that each side of a polygon is the side of exactly one other polygonal face.

(b) The polyhedron is said to be *convex* if the straight line segment connecting any two points in the polyhedron (possibly on its surface) is entirely within the polyhedron.

(c) \(F = 9 + m\)

(d) \(E = \frac{1}{2} (9 \times 4 + m \times 8) = 18 + 4m\).

(e) Euler’s formula says that for a convex polyhedron with \(V\) vertices, \(E\) edges, and \(F\) faces,

\[ V - E + F = 2. \]

(f) \(V = 2 + E - F = 2 + 18 + 4m - 9 - m = 11 + 3m\)

(g) Since \(2F \geq V + 4\) we obtain \(18 + 2m \geq 11 + 3m + 4\), which simplifies to

\[ m \leq 3. \]

Thus the number \(m\) of octagon faces is at most 3.

\[ \diamondsuit \]
2. (Exercise 10.1; 12 points.) Let \( a \) and \( b \) be two positive integers. Suppose we have the following chain of divisions:

\[
b = qa + 12, \quad a = \ell 12 + 5, \quad 12 = 2 \times 5 + 2, \quad 5 = 2 \times 2 + 1,
\]

where \( q \) and \( \ell \) are integers. (The numbers 12, 5, 2, 1 are the remainder terms.)

(a) What is the value of the highest common factor of \( a \) and \( b \)?

(b) Find integers \( r \) and \( s \) such that \( ra + bs = 1 \). (Write these integers in terms of the unknown \( q \) and \( \ell \).)

(c) Find \( \text{hcf}(17, 29) \). (This corresponds to \( q = \ell = 1 \).)

**Solution.**

(a) By the Euclidean algorithm, the highest common factor is the last nonzero division remainder. So \( \text{hcf}(a, b) = 1 \).

(b) From the given divisions, we obtain

\[
1 = 5 - 2 \times 2 = 5 - 2 \times (12 - 2 \times 5) = 5 \times 5 - 2 \times 12 = 5(a - \ell 12) - 2 \times 12 = 5a - (2 + 5\ell)12 = 5a - (2 + 5\ell)(b - qa).
\]

The last term in this chain of equalities simplifies to \([5 + (2 + 5\ell)q]a - (2 + 5\ell)b\). Therefore

\[
1 = [5 + (2 + 5\ell)q]a - (2 + 5\ell)b.
\]

(c) We immediately obtain \( \text{hcf}(17, 29) = 1 \).
3. **(Exercise 10.4; 16 points.)** Answer the following questions, where $a$ and $b$ are nonzero integers.

(a) State the mathematical definition of: *Integer $a$ divides integer $b$.*

(b) Prove from the definition the statement: *If integer $a$ divides integer $b$ and $b$ divides $a$ then $a = \pm b$."

(c) If $a, b, c, d$ are integers and $ab + cd = 1$, what is the highest common factor of $a$ and $c$? Explain your answer.

(d) If $n$ is an integer, find the highest common factor of $6n + 8$ and $4n + 5$. Explain your answer.

**Solution.**

(a) $a$ divides $b$ if there is an integer $m$ such that $b = am$.

(b) $a\mid b$ means that $b = am$ and $b\mid a$ means that $a = bn$ for integers $m$ and $n$. Thus

$$b = am = bnm.$$

And since $b \neq 0$, we have $nm = 1$. This is only possible if $n = m = \pm 1$. Therefore $b = \pm a$.

(c) $\text{hcf}(a, c) = 1$. In fact, let $h$ be this highest common factor. Then $a = rh$ and $c = sh$. Therefore

$$1 = ab + cd = (rb + sd)h$$

so $h$ must divide 1. But the only (positive) divisor of 1 is 1. So $\text{hcf}(a, c) = 1$.

(d) Note that

$$2(6n + 8) - 3(4n + 5) = 1.$$

Therefore

$$\text{hcf}(6n + 8, 4n + 5) = 1.$$
4. (Exercise 11.4; 20 points.) Given \(a, b \in \mathbb{N}\) and \(d = \text{hcf}(a, b)\), let \(\alpha, \beta\) be coprime integers such that \(a = \alpha d\) and \(b = \beta d\).

(a) State the mathematical definition of the highest common factor \(\text{hcf}(a, b)\).
(b) State the mathematical definition of the least common multiple \(\text{lcm}(a, b)\).
(c) State the mathematical definition of two integers \(\alpha, \beta\) being coprime to each other.
(d) Show that \(\text{lcm}(\alpha, \beta) = \alpha \beta\). (You may take for granted: if \(\alpha\) divides \(\beta n\), where \(n\) is an integer, then \(\alpha\) divides \(n\).)
(e) Given \(d, \ell \in \mathbb{Z}\), then \(\alpha d\) and \(\beta d\) divide \(\ell\) if and only if \(d\) divides \(\ell\) and \(\alpha\) and \(\beta\) divide \(\frac{\ell}{d}\). It follows that

\[
\text{lcm}(\alpha, \beta) = \alpha \beta d.
\]

Taking this easy to prove observation for granted (and using (d)), show that \(\text{lcm}(a, b)\text{hcf}(a, b) = ab\).

Solution.

(a) \(\text{hcf}(a, b)\) is the largest positive integer that divides both \(a\) and \(b\).
(b) \(\text{lcm}(a, b)\) is the smallest positive integer divisible by both \(a\) and \(b\).
(c) Integers \(a, b\) are coprime to each other if \(\text{hcf}(a, b) = 1\).
(d) From the definition of the least common multiple, there are \(m, n \in \mathbb{Z}\) such that

\[
\alpha m = \text{lcm}(\alpha, \beta) = \beta n.
\]

Thus \(\alpha\) divides \(\beta n\), implying that \(\alpha\) divides \(n\). Thus \(n = \alpha \ell\) for some integer \(\ell\). It follows that

\[
\text{lcm}(\alpha, \beta) = \beta n = \beta \alpha \ell.
\]

But \(\alpha \beta\) is a common multiple of \(\alpha\) and \(\beta\), therefore it must be the smallest.
(e) We now have

\[
\text{lcm}(a, b) = \text{lcm}(\alpha d, \beta d) = \text{lcm}(\alpha, \beta) d = \alpha \beta d = ab/d.
\]

We conclude that

\[
\text{lcm}(a, b)\text{hcf}(a, b) = ab.
\]
5. **(Exercise 11.8; 12 points.)** In the following questions, $x$ and $y$ are integers. Justify your answers.

(a) What is a *Diophantine equation*?

(b) If $x^2 - x = y^3$, explain why $x = a^3$ for some integer $a$.

(c) If $x^2 = y^4 - 77$, what are the possible (integer) values of $y$?

**Solution.**

(a) A Diophantine equation is an equation in several variables the solutions of which should be integer numbers.

(b) The equation can be written in the form $x(x - 1) = y^3$. Since $x$ and $x - 1$ are coprime to each other, and since the right-hand side is the cube of an integer, $x$ itself must be the cube of an integer by a proposition discussed in class.

(c) The equation can be written as $77 = (y^2 - x)(y^2 + x)$. This means that $y^2 - x$ can take as possible values 1, 7, 11, 77, and $y^2 + x$ can take the values 77, 11, 7, 1, respectively. This means that $2y^2 = (y^2 - x) + (y^2 + x)$ can take the values 78 or 18. Thus $y^2$ can be either 39 or 9. But 38 is not an integer square. We conclude that the only possible values of $y$ are $\pm 3$. $\diamondsuit$
6. (Exercise 8.9; 12 points.) Suppose $P(n)$ is a statement to be proved for all $n$.

(a) Explain succinctly what proving by induction that $P(n)$ is true for all $n \in \mathbb{N}$ amounts to.

(b) Let $0 < q < \frac{1}{2}$ and $P(n)$ the statement: $(1 + q)^n \leq 1 + 2^n q$. Show that if $P(n)$ is true, then $P(n + 1)$ is also true.

(c) Prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Solution.

(a) To prove $P(n)$ for all $n \geq 1$ by the induction method amounts to showing two things: (1) that the statement $P(1)$ is true and (2) that, for each $n \geq 1$, if $P(n)$ is true then $P(n + 1)$ is also true.

(b) Suppose that $(1 + q)^n \leq 1 + 2^n q$. Then

$$(1 + q)^{n+1} = (1 + q)(1 + q)^n \leq (1 + q)(1 + 2^n q).$$

Under the assumption $0 < q < \frac{1}{2}$, we have $1 + q \leq \frac{3}{2}$, so that (for $n \geq 1$)

$$1 + 2^n (1 + q) \leq 2^n \left( \frac{1}{2^n} + \frac{3}{2} \right) \leq 2^{n+1}.$$

This implies

$$(1 + q)(1 + 2^n q) = 1 + 2^n q + q + 2^n q^2 = 1 + [1 + 2^n (1 + q)] q \leq 1 + 2^{n+1} q.$$

We can thus conclude that

$$(1 + q)^{n+1} \leq 1 + 2^{n+1} q.$$

But this is the statement $P(n + 1)$. So $P(n)$ implies $P(n + 1)$

(c) All that is left to check is that $P(1)$ is true. But $P(1)$ is the statement

$$1 + q \leq 1 + 2q,$$

which is clearly true since $q \leq 2q$. 

\diamond