Math 310 – Fall 2019 – Exam I
1. (Exercise 2.3; 12 points.) For each of the following statements, indicate which are true and which are false. When true, give a proof (very short!); when false, give a counterexample to show it is false.

(a) The product of two rational numbers is always rational.
(b) The product of two irrational numbers is always irrational.
(c) The product of two irrational numbers is always rational.
(d) The product of a non-zero rational and an irrational is always irrational.

Solution.

(a) True: if \( \alpha = \frac{m}{n} \) and \( \beta = \frac{p}{q} \) are rationals, then \( \alpha \beta = \frac{mp}{nq} \) is also rational.

(b) False: let \( \alpha = \beta = \sqrt{2} \). Then \( \alpha \) and \( \beta \) are irrational but \( \alpha \beta = 2 \), which is rational.

(c) False: if \( \alpha = \beta = 2^{1/4} \), then \( \alpha \) and \( \beta \) are irrational (else \( \alpha^2 = \sqrt{2} \) would be rational by part (a)) but \( \alpha \beta = \sqrt{2} \) is irrational.

(d) True: if \( \alpha = \frac{m}{n} \) (\( m, n \neq 0 \)) and \( \beta \) is a number such that \( \alpha \beta = \frac{p}{q} \), then \( \beta = \frac{np}{mq} \) would be rational.
2. (Exercises 1.1, 1.5; 13 points.) Which of the following statements are true, and which are false? The first 8 items refer to the set

\[ A = \{\alpha, \{1, \alpha\}, \{3\}, \{\{1, 3\}\}, 3\}. \]

(It is not necessary to justify your answers.)

(a) \(\emptyset \subseteq A\)
(b) \(\{\{1, \alpha\}\} \subseteq A\)
(c) \(\{3, 3\} \subseteq A\)
(d) \(\{\alpha\} \notin A\)
(e) \(\{1, 3\} \in A\)
(f) \(\{1, \alpha\} \notin A\)
(g) \(\{\{1, 3\}\} \subseteq A\)
(h) \(\{1, \alpha\} \subseteq A\)
(i) \(n = 3\) if \(n^2 - 2n - 3 = 0\)
(j) \(n = 3\) only if \(n^2 - 2n - 3 = 0\)

Solution.

(a) True
(b) True
(c) True
(d) True
(e) False
(f) False
(g) False
(h) False
(i) False
(j) True
3. (Exercise 3.3; 12 points.) Of the following real numbers expressed in decimal form, which are rational, and which are irrational? (Give a brief justification.)

(a) $0.101001000100001000001\ldots$
(b) $1.b_1b_2b_3\ldots$, where $b_i = 1$ if $i$ is a square, and $b_i = 0$ if $i$ is not a square.
(c) $0.a_1a_2a_3\ldots$, were for $n = 1, 2, 3, \ldots$, the value of $a_n$ is the number $0, 1, 2, 3$ or $4$ which is the remainder on dividing $n$ by 5.
(d) Express the rationals among the above as $\frac{m}{n}$ with $m, n \in \mathbb{Z}$. (A short calculation is needed here.)

**Solution.**

(a) Irrational. This is because the decimal expression is not periodic.
(b) Irrational. This is because the decimal expression of $1.b_1b_2b_3\ldots = 1.10010001000001\ldots$ is not periodic.
(c) Rational. This is because the decimal expression of $0.a_1a_2a_3\ldots = 0.1234012340\ldots = 0.\overline{12340}$ is periodic.
(d) For the number in item (c) we have:

$$0.\overline{12340} = 12340 \left( \frac{1}{(10^5)^1} + \frac{1}{(10^5)^2} + \frac{1}{(10^5)^3} + \cdots \right) = 12340 \frac{1 - \frac{1}{10^5}}{1 - \frac{1}{10^5}} = 12340 \frac{1}{10^5 - 1} = \frac{12340}{99999}$$
4. (Exercise 5.2; 12 points.) Consider the inequality
\[ x^2 + x + 1 > \frac{x - 1}{2x - 1}. \]

(a) Show that this is equivalent to
\[ \frac{x^2(2x + 1)}{2x - 1} > 0. \]

(b) Find the inequality’s solution set.

Solution.

(a) Note that
\[ x^2 + x + 1 > \frac{x - 1}{2x - 1} \iff x^2 + x + 1 - \frac{x - 1}{2x - 1} > 0 \iff \frac{(x^2 + x + 1)(2x - 1) - (x - 1)}{2x - 1} > 0 \iff \frac{2x^3 + x^2}{2x - 1} > 0. \]

Therefore
\[ x^2 + x + 1 > \frac{x - 1}{2x - 1} \iff \frac{x^2(2x + 1)}{2x - 1} > 0. \]

(b) This inequality holds for the set of \( x \) such that \( x \neq 0 \) and: (i) both \( 2x + 1 > 0 \) and \( 2x - 1 > 0 \) or (ii) \( 2x + 1 < 0 \) and \( 2x - 1 < 0 \). In case (i), \( x > 1/2 \) and in case (ii), \( x < -1/2 \). Thus the solution set consists of the \( x \in \mathbb{R} \) such that
\[ x > 1/2 \text{ or } x < -1/2. \]

Equivalently, this is the set
\[ \{ x \in \mathbb{R} : |x| > 1/2 \}. \]
5. (Exercise 7.2; 12 points.) In the process of solving for the roots of $x^3 - 6x^2 + 13x - 12 = 0$, the first step is to eliminate the second degree term by a substitution of the form $y = x + c$, where $c$ is a real number. The resulting equation in $y$ then takes the form $y^3 + 3hy + k = 0$.

(a) What is the value of $c$?
(b) What is the value of $h$?
(c) What is the value of $k$?

It is not necessary to justify your answers, although some calculation is likely needed since I don’t assume or expect you’ll remember the formulas by heart.

**Solution.** In order to eliminate the second degree term, define $y = x - 2$. Then

$$y^3 = (x - 2)^3 = x^3 - 6x^2 + 12x - 8 = -13x + 12 + 12x - 8 = -x + 4 = -y - 2 + 4 = -y + 2$$

and the cubic equation in $y$ has the form

$$y^3 + y - 2 = 0.$$ 

Therefore

(a) $c = -2$
(b) $h = 1/3$
(c) $k = -2$
6. (Exercises 5.8, 4.7; 12 points.) Prove the following statements:

(a) Prove that if $x, y > 0$ then

$$\frac{x + y}{2} \geq \sqrt{xy}. $$

(b) Prove using (a) that among all rectangles with a given area $A$, the square of side $\sqrt{A}$ has the smallest perimeter.

**Solution.**

(a) As both sides of the inequality are non-negative, this is equivalent to proving that

$$\left( \frac{x + y}{2} \right)^2 \geq (\sqrt{xy})^2. $$

This can be seen as follows:

$$\left( \frac{x + y}{2} \right)^2 - (\sqrt{xy})^2 = \frac{x^2 + 2xy + y^2}{4} - xy = \frac{x^2 + 2xy + y^2 - 4xy}{4} = \frac{x^2 - 2xy + y^2}{4} = \frac{(x - y)^2}{4} \geq 0.$$ 

This is what needed to be shown.

(b) Let $P_r$ and $P_s$ denote the perimeter of a rectangle and a square, respectively, both figures having area $A$. Let $x$ and $y$ be the length of the base and height of the rectangle. Then $A = xy$ and the side-length of the square is $\sqrt{A} = \sqrt{xy}$. Thus $P_r = 2x + 2y$, $P_s = 4\sqrt{xy}$. Part (a) now allows us to conclude that

$$P_r = 2x + 2y \geq 4\sqrt{xy} = P_s.$$
7. (Exercise 6.4; 12 points.) Given complex numbers \( u, v \),

(a) Show that \( u \overline{v} + v \overline{u} \) is a real number.

(b) Show that \( u \overline{v} + v \overline{u} \leq 2|u||v| \).

(c) Show that \( |u + v|^2 \leq (|u| + |v|)^2 \).

(d) In order to conclude from part (c) the triangle inequality

\[ |u + v| \leq |u| + |v| \]

we need to be sure that if \( a \) and \( b \) are non-negative numbers such that \( a^2 \leq b^2 \), then \( a \leq b \). Why is this true? (Hint: start by writing \( b^2 - a^2 \) as a product of two terms.)

**Solution.** Let us write \( u \) and \( v \) in polar form: \( u = |u|e^{i\theta}, v = |v|e^{i\phi} \). Then

(a) \( u \overline{v} + v \overline{u} = |u| |v| (e^{i(\theta - \phi)} + e^{-i(\theta - \phi)}) = 2|u||v| \cos(\theta - \phi) \), which is a real number.

(b) Since \( \cos(\theta - \phi) \leq 1 \), we can conclude from (a) that \( u \overline{v} + v \overline{u} \leq 2|u||v| \).

(c) Note that, due to part (b),

\[ |u + v|^2 = (u + v) \overline{(u + v)} = u\overline{u} + 2uv + v\overline{v} \leq |u|^2 + 2|u||v| + |v|^2 = (|u| + |v|)^2. \]

(d) Note that \( 0 \leq b^2 - a^2 = (b + a)(b - a) \). If \( b = a = 0 \) then clearly \( b \geq a \). If \( b + a > 0 \), we may cancel out this factor to obtain \( b - a \geq 0 \), which is equivalent to \( b \geq a \).
8. (Exercise 6.7; 15 points.) Consider the following complex numbers: \( \omega = e^{i \frac{2\pi}{5}} \), \( \alpha = \omega + \omega^4 \), \( \beta = \omega^2 + \omega^3 \).

(a) What does it mean to say that \( \omega \) is a 5th root of unity?
(b) Show that \( \alpha = 2 \cos \frac{2\pi}{5} \).
(c) Show that \( 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0 \). (Hint: write \( x^5 - 1 \) as a product of \( x - 1 \) and a polynomial of degree 4.)
(d) Find the value of \( \alpha + \beta \)
(e) Find the value of \( \alpha \beta \)

Solution.

(a) To be a 5th root of unity means that \( \omega \) is a root of the equation \( x^5 = 1 \).
(b) Note that \( \omega^4 = \omega^{-1} = \overline{\omega} \). Then
\[
\alpha = \omega + \overline{\omega} = 2 \cos \frac{2\pi}{5}.
\]
(c) We have the factorization
\[
x^5 - 1 = (x - 1) \left( 1 + x + x^2 + x^3 + x^4 \right).
\]
Thus
\[
0 = \omega^5 - 1 = (\omega - 1) \left( 1 + \omega + \omega^2 + \omega^3 + \omega^4 \right).
\]
As \( \omega - 1 \neq 0 \) we may cancel this term to conclude that
\[
1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0.
\]
(d) It follows that
\[
\alpha + \beta = \omega + \omega^4 + \omega^2 + \omega^3 = -1
\]
and
(e)
\[
\alpha \beta = (\omega + \omega^4)(\omega^2 + \omega^3) = \omega^3 + \omega^4 + \omega^6 + \omega^7 = \omega^3 + \omega^4 + \omega + \omega^2 = -1.
\]