When asked to justify your statements (for example, in the True or False questions) by giving a proof or counter example, you may point to results from the book or homework problems. In such cases, be sure to state which Theorem/Proposition you are using, or which homework problem you are appealing to. (For example, Exercise number x of HW y, or Proposition x, y, z.) Or simply state the result you need. (Example: “by the Theorem that says that the continuous image of a connected set is connected . . . ”)

1. (10 points.) Let $X_1, X_2$ be topological spaces and give $X = X_1 \times X_2$ the product topology. Let $\pi_1, \pi_2$ be the coordinate projections. Prove or give a counter example to the following statement: Let $G$ be an open set in $X$ and let $U_i \subseteq X_i$ be open sets such that $U_i \subseteq \pi_i(G)$ for $i = 1, 2$. Then $U_1 \times U_2 \subseteq G$.

2. (10 points.) For $i = 1, 2, 3, \ldots$, give $X_i = \{0, 1\}$ the discrete topology and give $X = X_1 \times X_2 \times X_3 \times \cdots$ the product topology.
   (a) Is $X$ connected?
   (b) Is $X$ compact?
   (c) Is $X$ metrizable?
   (d) Is the product topology in this case the same as the discrete topology on $X$?

3. (10 points.) For $i = 1, 2, 3, \ldots$, give $X_i = \{0, 1\}$ the discrete topology and give $X = X_1 \times X_2 \times X_3 \times \cdots$ the product topology. Define the shift map $f : X \to X$ such that $y = f(x)$ if and only if $y_i = x_{i+1}$ for all $i$.
   We say that a point $x \in X$ is periodic if there is $n \in \mathbb{N}$ such that $f^n(x) = x$. Here $f^n = f \circ f \circ \cdots \circ f$ ($n$ times). Thus, for example, $x = (0, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)$ is periodic since $f^3(x) = x$.
   (a) Is $f$ continuous?
   (b) Is $f$ surjective?
   (c) Is $f$ injective?
   (d) Show that the set of periodic points is dense in $X$.

4. (10 points.) Let $X$ be an infinite set and $\mathcal{T}$ the family of subsets $U \subseteq X$ such that either $U$ is empty or $X \setminus U$ is finite (possibly empty).
   (a) Show that $\mathcal{T}$ is a topology on $X$.
   (b) Is $(X, \mathcal{T})$ a Hausdorff space?
   (c) Is $(X, \mathcal{T})$ compact?
(d) If \(X = \mathbb{R}\), what is the closure of the set \(\mathbb{N} = \{1, 2, 3, \ldots\}\)?

5. (10 points.) Let \((X, \rho)\) be a metric space. Determine whether each statement is True or False. Explain your answers with a proof or counterexample.

(a) If \(X\) is compact then every continuous function \(f : X \to \mathbb{R}\) is bounded.
(b) If every continuous function \(f : X \to \mathbb{R}\) is bounded then \(X\) is compact.
(c) If \(X\) is compact and \(f, f_1, f_2, \ldots\) are continuous functions such that \(f_n \to f\) then the convergence is uniform.
(d) If \(f, f_1, f_2, \ldots\) are bounded and continuous real valued functions on \(X\) such that each \(f_n\) is uniformly continuous and \(f_n \to f\) uniformly, then \(f\) is also uniformly continuous.

6. (10 points.) Let \(X\) be a set and \(\mathcal{T}_1, \mathcal{T}_2\) be two topologies on \(X\) such that \(\mathcal{T}_1 \subset \mathcal{T}_2\). Determine whether each item is True or False. If True give a proof, and if False, give a counterexample.

(a) If \((X, \mathcal{T}_1)\) is Hausdorff then \((X, \mathcal{T}_2)\) is also Hausdorff.
(b) If \((X, \mathcal{T}_1)\) is regular then \((X, \mathcal{T}_2)\) is also regular.

7. (10 points.) Let us define a topology \(\mathcal{T}\) on the set of real numbers \(\mathbb{R}\) such that a base for \(\mathcal{T}\) consists of the open intervals and sets of the form \(I \setminus K\) where \(I\) is an open interval and \(K = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}\). For each item below, justify your answer with a proof.

(a) Is \(\mathcal{T}\) Hausdorff?
(b) Is \(\mathcal{T}\) regular?
(c) Is \(\mathcal{T}\) completely regular?
(d) Is \(\mathcal{T}\) normal?

8. (10 points.) Determine whether each of the following statements is True or False. If True give a proof and if False provide a counterexample.

(a) Suppose \(U, V\) are subsets of a set \(X\). If \(U\) and \(V\) are disjoint, then \(X = (X \setminus U) \cup (X \setminus V)\).
(b) Let \(f : X \to Y\) be any function and \(A \subseteq Y\). Suppose \(f^{-1}(A) \subseteq U\) for some set \(U \subseteq X\). Then \(A \subseteq Y \setminus f(X \setminus U)\).
(c) If \(f : X \to Y\) is surjective and \(U, V\) are disjoint subsets of \(X\) then \(Y = f(X \setminus U) \cup f(X \setminus V)\).
(d) If \(f : X \to Y\) is surjective and \(U, V\) are disjoint subsets of \(X\) then \(Y \setminus f(X \setminus U)\) and \(Y \setminus f(X \setminus V)\) are disjoint.

9. (10 points.) Let \(X\) and \(Y\) be topological spaces. We say that \(f : X \to Y\) is closed if the image of every closed set is closed. Suppose \(f\) is a continuous, surjective, and closed function, and that \(X\) is a normal topological space. Show that \(Y\) is also normal.

10. (10 points.) A topological space \(X\) will be said to have the property \(E\) (a name I have just invented) if the following holds: for every closed subset of \(X\) and continuous function \(f : C \to \mathbb{R}\), there exists a continuous function \(F : X \to \mathbb{R}\) such that \(F(x) = f(x)\) for all \(x \in C\). Show that if \(X\) has the property \(E\) then it must be normal.