Homework set 2 – Solutions
Math 4171 – Renato Feres

1. Read pages 16 through 22 of the textbook.

2. Exercise 3, page 23. (20 points.) If a continuous function is constant on a dense subset, it is constant. More precisely, if \( f : (X, d) \rightarrow (Z, \rho) \) is continuous, \( A \) is a dense subset of \( X \), and \( z \in Z \) such that \( f(a) = z \) for every \( a \in A \), show that \( f(x) = z \) for every \( x \) in \( X \).

Solution. The set \( \{z\} \) is closed in \( Z \). (A single point set is always closed in a metric space. In fact, if \( z' \) is any other point, the open ball centered at \( z' \) with radius \( \rho(z, z') \) is entirely contained in the complement \( Z \setminus \{z\} \), so this complement is open.) Since \( f \) is continuous, the inverse image of \( \{z\} \) under \( f \) is a closed set (Theorem 1.3.3 (c)) contained \( A \). But \( A \) is dense, so \( X = \text{cl} A \subseteq \{x \in X : f(x) = z\} \).

This means that \( f \) is constant, equal to \( z \), on all of \( X \).

3. Exercise 4, page 23. (20 points.) If \( f : (X, d) \rightarrow (Z, \rho) \) is both continuous and surjective and \( A \) is a dense subset of \( X \), show that \( f(A) \) is a dense subset of \( Z \).

Solution. Let \( B = \text{cl} f(A) \) (the closure of the set \( f(A) \) in \( Z \)). We wish to show that \( B = Z \). Suppose, for a contradiction, that this is not the case and let \( z \in Z \setminus B \). Since \( Z \setminus B \) is open, there is a sufficiently small \( r > 0 \) such that the open ball \( B(z, r) \) is contained in \( Z \setminus B \). Consider the subset \( U \) of \( X \) defined as \( U = f^{-1}(Z \setminus B) \). Then \( U \) is open since \( f \) is continuous (Theorem 1.3.3 (b)) and it is non-empty since \( f \) is surjective. Since \( A \) is dense in \( X \), it must intersect \( U \) (this follows from Proposition 1.1.20). Let \( a \in A \cap U \). Then \( f(a) \in Z \setminus B \). But this is not possible since \( B \) contains \( f(A) \). This contradiction shows that \( f(A) \) is dense in \( Z \).

4. Exercise 10, page 23. (20 points.) Show that \( x^2 \) maps Cauchy sequences in \( \mathbb{R} \) into Cauchy sequences. (Hint: use Proposition 1.2.16 (c).)

Solution. Let \( x_1, x_2, \ldots \) be a Cauchy sequence in \( \mathbb{R} \). We need to show that \( x_1^2, x_2^2, \ldots \) is also a Cauchy sequence. We know from Proposition 1.2.16 (c) that Cauchy sequences constitute a bounded set. Thus \( |x_n| \leq c \) for all \( n \). For any given \( \epsilon > 0 \), let \( N \) be a sufficiently large positive integer such that \( |x_n - x_m| < \epsilon/(2c) \) for all \( n, m \geq N \). Therefore, for all \( n, m \geq N \),

\[
|x_n^2 - x_m^2| = |(x_n + x_m)(x_n - x_m)| \leq (|x_n| + |x_m|)|x_n - x_m| < 2c\epsilon/(2c) = \epsilon.
\]
This means that the sequence of squares is also Cauchy.

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5. **Exercise 11, page 23.**

(a) (10 points.) Give an example of two equivalent metrics on a set \( X \) that have different sets of uniformly continuous functions.

(b) (10 points.) Do the equivalent metrics \( \rho \) and \( d \) in Exercise 1.1.11 (page 10) have the same uniformly continuous functions?

**Solution.**

(a) Let \( X = (0, \infty) \) be given the standard metric \( d(x, y) = |x - y| \) as well as the new metric \( \rho(x, y) = \left| x^2 - y^2 \right| \).

(It is not difficult to check that \( \rho \) is, in fact, a metric.) We consider functions into \( Z = \mathbb{R} \) (where \( Z \) is given the standard absolute value metric). Consider \( f(x) = x^2 \). This function is not uniformly continuous with respect to \( d \). (This is shown just as in Example 1.3.16 (c).) However, with respect to \( \rho \), we have

\[
|f(x) - f(y)| = \left| x^2 - y^2 \right| = \rho(x, y).
\]

Thus for any \( \epsilon > 0 \), taking \( \delta = \epsilon \), we have: \( |f(x) - f(y)| < \epsilon \) for all \( x, y \) such that \( \rho(x, y) \leq \delta \). Therefore \( f \) is uniformly continuous on \( (X, \rho) \) but not on \( (X, d) \).

(b) Yes. To see this, first observe the following: for any two points \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) of \( X_1 \times X_2 \),

\[
\rho(x, y) = \max(d_1(x_1, y_1), d_2(x_2, y_2)) \quad \text{and} \quad d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)
\]

are related by the inequalities

\[
\rho(x, y) \leq d(x, y) \leq 2\rho(x, y).
\]

If \( f : X_1 \times X_2 \to W \) is uniformly continuous for the metric \( d \) on \( X_1 \times X_2 \) (let us suppose \( W \) has metric \( \sigma \), then given any \( \epsilon > 0 \), there is \( \delta > 0 \) such that \( \sigma(f(x), f(y)) < \epsilon \) whenever \( d(x, y) < \delta \). Therefore, whenever

\[
\rho(x, y) < \delta/2,
\]

we have \( d(x, y) \leq 2\rho(x, y) < 2\delta/2 = \delta \) hence \( \sigma(f(x), f(y)) < \epsilon \). This means that \( f \) is uniformly continuous with respect to \( \rho \). Similarly, using the inequality \( \rho \leq d \), we can show that if a function \( f \) is uniformly continuous relative to the metric \( \rho \), it must also be absolutely continuous relative to \( d \). (Here we use the same \( \delta \).) This shows that the two metrics define the same notion of uniform continuity.

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6. **Exercise 13, page 23.** (20 points.) Proposition 1.3.7 says, in effect, that if \( (X, d) \) is a metric space and \( C(X) \) is the set of all continuous functions \( f : X \to \mathbb{R} \), then \( C(X) \) is a vector space over \( \mathbb{R} \). Show that \( C(X) \) is a finite-dimensional vector space if and only if \( X \) is a finite set. (Hint: use Urysohn's lemma.)

**Solution.** If \( X \) is a finite set, then the space of continuous functions \( C(X) \) consists of all functions \( f : X \to \mathbb{R} \). (Every subset of \( X \) is both open and closed.) This space of all functions is a finite dimensional vector space which can be identified with \( \mathbb{R}^n \) where \( n \) is the number of elements of \( X \). In fact, if we order the elements of \( X \) as \( \{x_1, \ldots, x_n\} \), the correspondence

\[
f \in C(X) \mapsto \{f(x_1), \ldots, f(x_n)\} \in \mathbb{R}^n
\]

is easily shown to be an isomorphism of vector spaces.
Now suppose $X$ is infinite. For any given positive integer $n$, let $x_1, \ldots, x_n$ be $n$ distinct elements of $X$. Let $r$ be the smallest distance among all the pairs: $d(x_i, x_j)$ for $i \neq j$. For each $i$, let $F_i$ be the closed ball of radius $r/4$ centered at $x_i$ and $G_i$ the open ball of radius $r/2$ centered at $x_i$. Notice that $G_i \cap G_j = \emptyset$ for all $i \neq j$.

Corollary 1.3.10 of Urysohn’s Lemma implies the existence of a continuous function $f_i : X \to \mathbb{R}$ taking values in $[0,1]$ such that $f_i$ is constant equal to 1 on $F_i$ and constant equal to 0 on the complement of $G_i$. I claim that the functions $f_1, \ldots, f_n$ are all linearly independent. This will show that the dimension of $C(X)$ is greater than or equal to $n$ for all positive integer $n$, so this dimension cannot be finite.

To prove linear independence of the $f_i$, suppose

$$a_1 f_1(x) + \cdots + a_n f_n(x) = 0$$

holds for all $x \in X$ and fixed numbers $a_1, \ldots, a_n$. Evaluating this expression at $x_j$ gives

$$0 = a_1 f_1(x_j) + \cdots + a_n f_n(x_j) = a_j$$

since $f_j(x_j) = 1$ and $f_i(x_j) = 0$ if $i \neq j$. As all the $a_i$ are zero, we conclude that we can find a linearly independent set of $n$ functions for every $n$. This shows that $C(X)$ is infinite dimensional when $X$ is an infinite set.

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