Homework set 3 – Solutions
Math 4171 – Renato Feres

1. **Read Section 1.4, pages 24-28, of textbook.**

2. **Exercise 3, page 29.** Show that the closure of a totally bounded set is totally bounded.

   **Solution.** Let $S$ be a totally bounded set in the metric space $(X, d)$. By the definition of a set being totally bounded, for any given $\epsilon > 0$ there are points $x_1, \ldots, x_n \in S$ such that $S \subseteq \bigcup_{k=1}^{n} B(x_k; \epsilon/2)$.

   Let $x \in \text{cl}S$ and choose $x' \in S$ such that $d(x, x') < \epsilon/2$. (Recall that the points in the closure are at a distance 0 from $S$, so such $x'$ exists.) Now choose $x_k$ such that $d(x', x_k) < \epsilon/2$. By the triangle inequality,
   
   
   
   This means that $\text{cl}S \subseteq \bigcup_{k=1}^{n} B(x_k; \epsilon)$, which is what we needed to show.

3. **Exercise 4, page 29.** Show that a totally bounded set is bounded. Is the converse true?

   **Solution.** Suppose that $S$ is totally bounded in the metric space $(X, d)$. By definition, there are finitely many points $x_1, \ldots, x_n \in S$ such that $S \subseteq \bigcup_{k=1}^{n} B(x_k; 1)$. Let $D$ be the maximum among all $d(x_i, x_j)$, $i, j \in \{1, \ldots, n\}$.

   Given any two points $x, y \in S$, we can find $i, j \in \{1, \ldots, n\}$ such that $d(x, x_i) < 1$ and $d(y, x_j) < 1$. By the triangle inequality (applied twice),
   
   Therefore the diameter of $S$ (which is the supremum of $d(x, y)$ over all points $x, y \in S$) is no greater than $D + 2$. Thus we conclude that $S$ is bounded.

   The converse is not true, as Exercise 11, page 29 (assigned below) shows.

4. **Exercise 7, page 29.**

   (a) If $G$ is an open set in a metric space $(X, d)$ and $K$ is a compact set with $K \subseteq G$, show that there is a $\delta > 0$ such that
   
   \[ \{ x : \text{dist}(x, K) < \delta \} \subseteq G. \]

   (b) Find an example of an open set $G$ in a metric space $X$ and a closed, noncompact subset $F$ of $G$ such that there is no $\delta > 0$ with $\{ x : \text{dist}(x, F) < \delta \} \subseteq G$.

   **Solution.**
(a) For each \( x \in K \) define

\[
F(x) = \text{dist}(x, X \setminus G).
\]

Note that \( F(x) > 0 \) for each \( x \in K \) since points of \( K \) are interior points of \( G \), hence at a positive distance from \( X \setminus G \). We know that \( F: K \to (0, \infty) \) is a continuous function by Corollary 1.3.6. By Corollary 1.4.3, there is \( a \in K \) such that \( 0 < F(a) = \inf \{ F(x) : x \in K \} \). (That is, the infimum is attained at some point in \( K \).) Let us set \( \delta = F(a)/2 \). Let \( x \in X \) be such that \( \text{dist}(x, K) < \delta \) and \( k \) any point in \( K \). Then

\[
\text{dist}(k, X \setminus G) \leq d(k, x) + \text{dist}(x, X \setminus G).
\]

Taking the infimum over \( k \in K \), we obtain

\[
2\delta = F(a) \leq \text{dist}(x, K) + \text{dist}(x, X \setminus G) < \delta + \text{dist}(x, X \setminus G).
\]

Therefore, \( \text{dist}(x, X \setminus G) > \delta > 0 \). This means that \( x \in G \). Therefore,

\[
\{ x \in X : \text{dist}(x, K) < \delta \} \subseteq G.
\]

(b) Let \( X = \{ (x, y) : x \geq 0 \} \) with the standard (Euclidean) metric. Let \( G \) be the open set defined by

\[
G = \{ (x, y) : y \in \mathbb{R}^2 : x > 0 \}.
\]

Let \( F = \{ (x, 0) : x \geq 0 \} \). It is not difficult to check that \( F \) is a closed set in \( X \) and \( G \) is an open subset of \( X \) containing \( F \). Then, for any given \( \delta > 0 \), we can find points in \( X \setminus G \) at a distance less than \( \delta \) from \( F \).

\[\diamondsuit\]

5. **Exercise 8, page 29.** If \((X_1, d_1), (X_2, d_2)\) are metric spaces, show that \(X_1 \times X_2\) is compact if and only if both \(X_1\) and \(X_2\) are compact.

**Solution.** Suppose \(X_1\) and \(X_2\) are compact and let \(x_1, x_2, \ldots\) be a sequence in \( X_1 \times X_2 \). Let us write \( x_n = (x_{n_1}^1, x_{n_2}^2) \). Then by Theorem 1.4.5, compactness of \(X_1\) implies that the sequence \(x_{1_1}^1, x_{2_1}^1, \ldots\) has a convergent subsequence \(x_{n_1}^1, x_{n_2}^1, \ldots\). Similarly, the sequence \(x_{1_2}^2, x_{2_2}^2, \ldots\) must have a convergent subsequence, \(x_{n_1}^2, x_{n_2}^2, \ldots\). Therefore \(x_{n_1}^1, x_{n_2}^2, \ldots \to x^1\) and \(x_{m_1}^1, x_{m_2}^2, \ldots \to x^2\). Thus the sequence

\[
(x_{n_1}^1, x_{n_2}^2), (x_{m_1}^1, x_{m_2}^2), \ldots \to (x^1, x^2).
\]

This shows (by Theorem 1.4.5) that \(X_1 \times X_2\) is compact.

For the converse, suppose \(X_1 \times X_2\) is compact, let \(x_{1}^1, x_{2}^1, \ldots\) be a sequence in \(X_1\) and let \(x^2\) be any element in \(X_2\). Then \((x_1^1, x^2), (x_2^1, x^2), \ldots\) is a sequence in \(X_1 \times X_2\), which must, by Theorem 1.4.5 have a convergent subsequence,

\[
(x_{n_1}^1, x^2), (x_{n_2}^1, x^2), \ldots \to (x^1, x^2).
\]

Thus \(x_{1}^1, x_{1}^2, \ldots\) has a convergent subsequence, so \(X_1\) must be compact. A similar argument holds for \(X_2\).

\[\diamondsuit\]
6. **Exercise 11, page 29.** Consider the metric space $\ell^\infty$ (see Exercise 1.1.12) and show that

$$
S = \left\{ x = \{x_n\} \in \ell^\infty : \sup_n |x_n| \leq 1 \right\}
$$

is not totally bounded and, therefore, not compact.

**Solution.** Suppose, for a contradiction, that $S$ is totally bounded. Thus (choosing $\varepsilon = 1/2$) we can find finitely many points $x_1, \ldots, x_n \in S$ such that

$$
S \subseteq \bigcup_{k=1}^{n} B(x_n;1/2).
$$

Consider now the elements $e_1 = (1,0,0,\ldots), e_2 = (0,1,0,0,\ldots), \ldots$ in $S$. Since there are only finitely many balls $B(x_n;1/2)$ and infinitely many of the $e_1, e_2, \ldots$, there must be (by the pigeon-hole principle) two of these, say $e_k, e_l$ for $k \neq l$, belonging to the same $B(x_n;1/2)$. Then, denoting by $x(k)$ the $k$th element of a sequence $x \in \ell^\infty$, we have:

$$
1 = |1 - 0| = |e_k(k) - e_l(k)| \leq |e_k(k) - x_n(k)| + |x_n(k) - e_l(k)| \leq d(e_k, x_n) + d(x_n, e_l) < \frac{1}{2} + \frac{1}{2} = 1.
$$

But this is a contradiction, which proves that $S$ is not totally bounded (hence not compact).