Homework set 5 - due 10/25/20
Math 4171 – Renato Feres

1. Read Section 1.6, pages 36 and 37. The Baire Category Theorem.

2. Exercise 2, page 37 of the textbook. Say that a set $E$ in a metric space is nowhere dense if $\text{int}(\text{cl}E) = \emptyset$. (The interior of the set's closure is empty.) If $(X,d)$ is a complete metric space and $A = \bigcup_{n=1}^{\infty} E_n$, where each $E_n$ is nowhere dense, show that $X \setminus A$ is dense in $X$.

3. Exercise 4, page 38 of the textbook. Show that a closed interval in $\mathbb{R}$ cannot be written as a countable union of closed subsets that are pairwise disjoint.

4. Read Chapter 2, Section 2.1, pages 39 to 43 of the textbook.

5. Exercise 7, page 43 of the textbook. For the moment, disregard the agreement at the start of Section 2.1 of the textbook that all topological spaces are Hausdorff. Show that a topological space $(X,T)$ is Hausdorff if and only if for any two distinct points $x$ and $y$ there is an open set $G$ such that $y \in G$ and $x \notin \text{cl}G$.

6. Exercise 8, page 44 or the textbook. If $A$ is a subset of $X$ and $L$ is the set of limit points of $A$, prove that any limit point of $L$ is a limit point of $A$. Is $L$ a closed set?

7. Exercise 10, page 44 or the textbook. Let $X$ denote the set of all sequences of real numbers $\{x_n : n \in \mathbb{N}\}$, and let $\mathcal{T}$ consist of all subsets $G$ of $X$ satisfying the following condition: for each $x = \{x_n\}$ in $G$ there are integers $n_1 < \cdots < n_N$ and an $\epsilon > 0$ such that

$$\{y = \{y_n\} \in X : |x_{n_k} - y_{n_k}| < \epsilon \text{ for } 1 \leq k \leq N\} \subset G.$$ 

(a) Show that $(X,\mathcal{T})$ is a topological space.

(b) Is there a metric on $X$ such that $\mathcal{T}$ is the collection of open sets for this metric? (For inspiration, look at Theorem 2.6.6, page 63 of the textbook.)