Homework set 6 – Solutions
Math 4171 – Renato Feres

1. Read Section 2.2, pages 44 to 46 of the textbook.

2. Exercise (2), page 46 of the textbook. Prove that the topology generated by a subbase is the intersection of all the topologies that contain it.

   Solution. First let us observe that if \( \{ T_\alpha : \alpha \in A \} \) is a family of topologies on a set \( X \) such that each \( T_\alpha \) contains a subbase \( S \), then the intersection \( T = \bigcap_{\alpha \in A} T_\alpha \) is also a topology that contains \( S \). In fact,
   
   (a) \( \emptyset, X \) are contained in each \( T_\alpha \), so they are contained in the intersection.
   
   (b) If \( G_i, i \in I \), are sets in \( T \), then each \( G_i \) lies in every \( T_\alpha \), hence the union \( \bigcup_{i \in I} G_i \) must also lie in every \( T_\alpha \) (by the axiom of topologies concerning unions). Hence \( \bigcup_{i \in I} G_i \) lies in \( T \).
   
   (c) If \( G_1, \ldots, G_n \) are elements of \( T \) then each \( G_i \) must belong to every \( T_\alpha \). Therefore, \( G_1 \cap \cdots \cap G_n \) must also lie in all \( T_\alpha \) by the axiom of topologies concerning finite intersections. Hence \( G_1, \ldots, G_n \) lies in \( T \).

   Therefore, \( T \) is also a topology.

   Now let \( T' \) be the topology generated by \( S \). Since \( T' \) is a topology that contains \( S \), it must be the case that \( T \subseteq T' \). On the other hand, let the set \( G \) be an element of \( T' \). Then \( G \) is the union of sets \( B_j, j \in J \), where each \( B_j \) is a finite intersection of sets in \( S \). But every topology containing \( S \) must also contain finite intersections of elements of \( S \) and arbitrary unions of such finite intersections. Therefore \( T \) must contain \( G \). We conclude that \( T = T' \), which is what we were asked to show.

3. Exercise 4, page 46 of the textbook. Consider the plane \( \mathbb{R}^2 \), and define an open half-plane to be a set of the form

   \( \{(x, y) \in \mathbb{R}^2 : ax + by < c\} \)

   for some choice of constants \( a, b, c \).

   (a) If \( a, b, c \in \mathbb{R} \), show that \( \{(x, y) \in \mathbb{R}^2 : ax + by > c\} \) is an open half-plane.

   (b) Show that the collection of all open half-planes is a subbase for the usual topology of \( \mathbb{R}^2 \).

   Solution.

   (a) Clearly, \( \{(x, y) \in \mathbb{R}^2 : ax + by > c\} = \{(x, y) \in \mathbb{R}^2 : (-a)x + (-b)y < -c\} \), which is an open half-plane.

   (b) We first need to verify that the collection of open half-planes is a subbase. First observe that \( \{(x, y) : x > -1\} \) and \( \{(x, y) : x < 1\} \) are half-planes whose union is all of \( \mathbb{R}^2 \). So (a) the union of all half-planes is \( \mathbb{R}^2 \). (b) Let \( (a, \beta) \neq (a', \beta') \). Then either \( a \neq a' \), or \( \beta \neq \beta' \) or both. Suppose \( a < a' \) (the other possibilities, \( a > a' \),
4. Read Section 2.3, pages 48 to 51 of the textbook.

5. Exercise 3, page 52 of textbook; modified. Suppose \((X, \mathcal{T})\) and \((W, l\ell)\) are topological spaces and \(\{A_i : i \in I\}\) is a collection of open sets in \(X\) such that \(X = \bigcup_i A_i\). Suppose that for each \(i\) there is a continuous function \(g_i : A_i \to W\) such that \(g_i(x) = g_j(x)\) when \(x \in A_i \cap A_j\), and let \(f : X \to W\) be defined by \(f(x) = g_i(x)\) when \(x \in A_i\). Show that \(f\) is continuous.

**Solution.** Let \(U\) be an open set in \(W\). We need to show that \(f^{-1}(U)\) is open in \(X\). Observe that

\[
f^{-1}(U) = \{x \in X : f(x) \in U\} = \bigcup_i \{x \in A_i : f(x) \in U\} = \bigcup_i \{x \in A_i : g_i(x) \in U\} = \bigcup_i g_i^{-1}(U).
\]

Now, each \(g_i^{-1}(U)\) is open in \(A_i\) in the subspace topology. This means that \(g_i^{-1}(U) = V_i \cap A_i\) where \(V_i\) is open in \(X\). Since \(A_i\) is also open in \(X\), their intersection \(g_i^{-1}(U)\) must be open in \(X\) for each \(i\). But \(f^{-1}(U)\), being the union of the \(g_i^{-1}(U)\), must therefore be open. Since \(U\) was arbitrary, we conclude that \(f\) is continuous.
6. **Exercise 4, page 52 of textbook.** If \((X_k, \mathcal{T}_k)\) is a topological space for \(1 \leq k \leq n\) and \(X = X_1 \times \cdots \times X_n\), show that \(\{\pi_k^{-1}(G_k) : 1 \leq k \leq n\) and \(G_k \in \mathcal{T}_k\}\) is a subbase for the product topology on \(X\).

**Solution.** Let \(\mathcal{S}\) be the collection of all subsets of \(X\) of the form \(\pi_k^{-1}(G_k)\) for \(G_k \in \mathcal{T}_k\) and \(1 \leq k \leq n\). Since \(X = \pi_k^{-1}(X_k)\) for each \(k\), \(X\) is already in this collection (in particular, the union of these sets is \(X\)).

Now, let \(x, y\) be two distinct points in \(X\). Since they are different, there must be a \(k\) so that their \(k\)th coordinate projections are different: \(\pi_k(x) \neq \pi_k(y)\). By the Hausdorff property, there are disjoint open sets \(U, V \in \mathcal{T}_k\) so that \(\pi_k(x) \in U\) and \(\pi_k(y) \in V\). Therefore \(\pi_k^{-1}(U)\) and \(\pi_k^{-1}(V)\) are disjoint sets in \(\mathcal{S}\) such that \(x \in \pi_k^{-1}(U)\) and \(y \in \pi_k^{-1}(V)\). This shows that \(\mathcal{S}\) is a subbase.

It remains to check that the subbase \(\mathcal{S}\) generates the product topology on \(X\). We know (Proposition 2.3.5) that the product topology on \(X\) is generated by the base \(\mathcal{B}\) consisting of rectangles \(G_1 \times \cdots \times G_n\) where \(G_k \in \mathcal{T}_k\) for \(k = 1, \ldots, n\). But it is clear that these rectangles are (finite) intersections of elements of \(\mathcal{S}\):

\[
G_1 \times \cdots \times G_n = \bigcap_{k=1}^{n} \pi_k^{-1}(G_k).
\]

Therefore \(\mathcal{B}\) is the base generated by the subbase \(\mathcal{S}\), hence the product topology is generated by \(\mathcal{S}\).

\(\diamondsuit\)

7. **Exercise 7, page 52 of textbook, modified.** If \(X_k\) and \(Z_k\) are topological spaces for \(1 \leq k \leq n\), \(X = X_1 \times \cdots \times X_n\), and \(Z = Z_1 \times \cdots \times Z_n\), show that \(X\) and \(Z\) are homeomorphic if \(X_k\) and \(Z_k\) are homeomorphic for \(1 \leq k \leq n\).

Note: the original problem claims that the converse is also true: If the product spaces are homeomorphic, then after possibly permuting the factors, the \(X_k\) and \(Z_k\) are homeomorphic. But I don't see how that can be true. See, for example, *Homeomorphisms of Product Spaces* by V. Trnková, Russian Math. Surveys 34:6 (1979), 144-160.

**Solution.** Let \(f_k : X_k \to Z_k\) be a homeomorphism for each \(k\). Define \(F : X \to Z\) so that

\[
F(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n)).
\]

First note that \(F\) is a bijection. It is surjective since

\[
F \left( f_1^{-1}(z_1), \ldots, f_n^{-1}(z_n) \right) = z
\]

for any given \(z = (z_1, \ldots, z_n) \in Z\); and it is injective since for any given \(x, y \in X\) such that \(F(x) = F(y)\),

\[
f_k(x_k) = \pi_k(F(x)) = \pi_k(F(y)) = f_k(y_k),
\]

which implies \(x_k = y_k\) for each \(k\) since \(f_k\) is a homeomorphism. Thus \(x = y\).

I claim that \(F\) is a homeomorphism. To be a little more explicit about all the maps involved in the definition \(F\), we can characterize \(F\) by

\[
\pi_k \circ F = f_k \circ \pi_k
\]

for all \(k\), where I'm using the notation \(\pi_k\) for the coordinate projections of both \(X\) and \(Z\).

Note that if \(G_k\) is open in \(Z_k\), then

\[
F^{-1}(\pi_k^{-1}(G_k)) = \pi_k^{-1}(f_k^{-1}(G_k))
\]

is an open set since \(f_k \circ \pi_k\) is continuous (by Propositions 2.3.3 and 2.3.5). But the \(\pi_k^{-1}(G_k)\) are the sets of a subbase that generates the topology of \(Z\), so \(F\) is continuous. (Proposition 2.3.2 (e)).
It remains to show that $F^{-1}$ is also continuous. But

$$F^{-1}(z) = (f^{-1}_1(z_1), \ldots, f^{-1}_n(z_n))$$

so the argument used to show that $F$ is continuous equally applies to $F^{-1}$. Therefore $F$ is a homeomorphism.

\[ \diamond \]