Homework set 8 - due 11/15/20
Math 4171 – Renato Feres

1. Read Section 2.5, pages 57 to 61.

2. Exercise 2, page 61 of the textbook. The following subset of $\mathbb{R}^2$ is often called the comb (see Figure 1):

$$C = ([0] \times [0, 1]) \cup \{(n^{-1}, y) : n \in \mathbb{N}, y \in [0, 1]\} \cup ([0, 1] \times \{0\}).$$

Show that the comb is connected. Is it pathwise connected?

Figure 1: The topologist’s comb.

Solution. Define $E_n = ([0,1] \times \{0\}) \cup \{(1/n, y) : y \in [0, 1]\}$ for $n \in \mathbb{N}$ and $E_0 = ([0,1] \times \{0\}) \cup ([0] \times [0, 1])$. Each of these sets is the union of a vertical tooth of the comb and the horizontal line segment. So these sets are connected since each is the union of two intersecting intervals (in $\mathbb{R}^2$). Thus we have a family of connected sets that have pairwise nonempty intersections. (The intersection is the horizontal line segment.) The union $C = E_0 \cup \bigcup_{n=1}^{\infty} E_n$ is therefore connected.

$C$ is indeed pathwise connected. Having in mind Proposition 2.5.6 (c) (the union of mutually intersecting pathwise connected sets is pathwise connected), it is enough to prove that each $E_n, E_0$ is pathwise connected. By the same proposition, it is enough to show that each interval $\{a\} \times [0, 1]$ or $[0, 1] \times \{b\}$ is pathwise connected. But this is clear since given any two points $x, y \in [0, 1]$ (say, $x \leq y$), the path $t \mapsto (1-t)x + ty, 0 \leq t \leq 1$, is a path in $[0, 1]$ from $x$ to $y$. 

\[\Diamond\]

3. Exercise 4, page 61 of the textbook. Prove Proposition 2.5.8. More specifically, prove the following statements. Let $X$ be a topological space. Show the following:

(a) (7 points.) Every pathwise connected set is contained in a pathwise connected component.
(b) (7 points.) Distinct pathwise connected components are disjoint.

(c) (6 points.) The union of all the pathwise connected components is the entire space $X$.

Solution.

(a) Let $E$ be a (nonempty) pathwise connected set. Let $\mathcal{F}$ be the collection of all pathwise connected sets containing $E$. By Proposition 2.5.6 (c), the union $F$ of all the sets in $\mathcal{F}$ is pathwise connected. It must also be maximal (for the partial order defined by set inclusion): if $G$ is a pathwise connected set containing $F$, then it also contains $E$, and therefore is in $\mathcal{F}$. Thus $G \subseteq F$ and, consequently, $G = F$.

(b) Let $F$ and $G$ two distinct pathwise connected components. If $F \cap G \neq \emptyset$ then, by Proposition 2.5.6 (c), the union $F \cup G$ would be a pathwise connected set properly containing $F$ and $G$. But this would contradict the maximality of $F$ and $G$. Therefore $F \cap G = \emptyset$.

(c) For each $x \in X$, the single point set $\{x\}$ is pathwise connected, so by part (a) of this exercise it must be contained in a pathwise connected component. Since this is true for every $x \in X$, the union of all the pathwise connected components must be equal to $X$.

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4. Exercise 7, page 61 of the textbook. Give an example of a pathwise connected space that is not locally pathwise connected. (A good drawing with sufficiently clear explanations is enough here.)

Solution. The following example is a variation on the topologist’s sine curve. See Figure 2. It consists of the union of three sets: (1) graph of $\sin(1/x)$ for $x \in (0, a)$ for some $a > 0$; (2) the single point set $\{(0,1)\}$; (3) and a path from the right-hand endpoint of the graph to $(0,1)$ that does not intersect the graph at any other points than the end point $(a, \sin(1/a))$.

This subset of $\mathbb{R}^2$ with the subspace topology is pathwise connected, since it can be written as the union of two paths that intersect at one point. But it is not locally pathwise connected. In fact, any neighborhood of the point $(0, 1)$ contained in the intersection of the space with an open ball center of radius less than, say $1/2$, will contain infinitely many disconnected components.

![Figure 2: A variation of the topologist’s sine curve, showing a small neighborhood of the point (0, 1).](image)

Although not necessary for the exercise, we could make this example more explicit by choosing $a = 2/\pi$ and the connecting path from the point $(0, 1)$ to the end of the sine curve as the graph of the function $f(x) = x \left(\frac{x}{2} - x\right) + 1$, for $0 \leq x \leq 2/\pi$.

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5. **Read section 2.6, up to page 63 (including Theorem 2.6.6).**

6. **Exercise 1, page 66 of the textbook. (Slightly modified here.)** Show that if \( X = \prod_i X_i \) and \( y = \{y_i\} \in X \), then for each index \( j \) the map \( \phi : X_j \to X \) defined by

\[
\phi(x)_j = \begin{cases} 
  x & \text{when } i = j \\
  y_i & \text{when } i \neq j 
\end{cases}
\]

is a homeomorphism of \( X_j \) onto \( \phi(X_j) \).

**Solution.** We need to show that \( \phi \) is a continuous bijection whose inverse is also continuous.

(a) \( \phi \) is injective: If \( \phi(x) = \phi(x') \) then \( x = \phi(x)_j = \phi(x')_j = x' \).

(b) \( \phi \) is surjective: If \( y \in X \), then \( \phi(y_j) = y \).

(c) \( \phi \) is continuous: Note that \( x \mapsto (\pi_i \circ \phi)(x) = y_i \) is the constant map for \( i \neq j \) and \( x \mapsto (\pi_j \circ \phi)(x) = x \) is the identity map. The constant map and the identity map are always continuous. Therefore, by Proposition 2.6.5, \( \phi \) must be continuous.

(d) The inverse of \( \phi \) is continuous: The inverse of \( \phi \) is the coordinate projection \( \pi_j \), which is continuous in the product topology.

\[ \diamond \]

7. **Exercise 2, page 66 of the textbook.** Let \( (X, d) \) be a metric space, and let \( C(X) \) be the set of all continuous functions from \( X \) into \( \mathbb{R} \). Show that the weak topology defined on \( X \) by the functions in \( C(X) \) is the given topology on \( X \) defined by the metric.

**Solution.** Let us write \( \mathcal{D} \) for the metric topology on \( X \), and \( \mathcal{W} \) for the weak topology on \( X \) defined by \( C(X) \). For each each \( x \in X \) let us define the function \( d_x : X \to \mathbb{R} \) by \( d_x(y) = d(x, y) \). Then \( d_x \) is a continuous real valued function on \( X \). (Note: an application of the triangle inequality gives \( |d_x(y) - d_x(y')| \leq d(y, y') \), which implies continuity of \( d_x \).) Now, for each \( r > 0 \),

\[
d_x^{-1}((-r, r)) = \{y \in X : d(x, y) < r\} = B(x; r).
\]

Thus the subbase for the weak topology on \( X \) defined by \( C(X) \) includes the open balls. This means that \( \mathcal{D} \subseteq \mathcal{W} \).

Now every element \( G \) in the subbase for the weak topology is of the form \( G = f^{-1}(U) \) where \( U \) is open in \( \mathbb{R} \) and \( f \in C(X) \). But we know that such \( G \) must be open in the metric topology. (The metric topology is the topology with respect to which the functions in \( C(X) \) are continuous.) Therefore the subbase for the weak topology is contained in \( \mathcal{D} \). This implies \( \mathcal{W} \subseteq \mathcal{D} \). We conclude that \( \mathcal{D} = \mathcal{W} \).