Homework set 9 - due 11/22/20

Math 4171 – Renato Feres

1. **Read the remaining parts of Section 2.6, pages 65 and 66.**

2. **Exercise 5, page 67 of the textbook, modified.** Let \((X_i, \mathcal{T}_i) : i \in I\) be a collection of topological spaces, and let \(X = \prod_i X_i\) have the product topology.
   
   (a) Show that if \(X\) is separable, then \(X_i\) is separable for each \(i \in I\).
   
   (b) Suppose \(I\) is finite, so that \(X = X_1 \times \cdots \times X_n\). If each \(X_i\) is separable, show that \(X\) is separable.
   
   (c) Show that if \(I\) is countable and \(X_i\) is separable for each \(i \in I\), then \(X\) is separable.

Note 1: in the original statement, one is to prove that \(X\) is separable if and only if \(I\) is countable and each \(X_i\) is separable. But this does not seem to be true. See, for example, *Product of separable spaces* by K.A. Ross and A.H. Stone in *The Mathematical Monthly*, April 1964, Vol. 71, No. 4, pp. 398-403.

Note 2: You may take for granted the facts proved in the appendix to the textbook that the product of finitely many countable sets is countable, and the union of countably many countable sets is countable.

3. **Exercise 6, page 67 of the textbook.** Let \((X_i, \mathcal{T}_i) : i \in I\) be a collection of topological spaces, and let \(X = \prod_i X_i\) have the product topology. If, for each \(i \in I\), \(C_i\) is a component of \(X_i\), is \(C = \prod_{i \in I} C_i\) a component of \(X\)?

4. **Exercise 7, page 67 of the textbook, modified.** Let \((X_i, \mathcal{T}_i) : i \in I\) be a collection of topological spaces, and let \(X = \prod_i X_i\) have the product topology.

   (a) Show that if \(X\) is locally connected, then each \(X_i\) is locally connected. (Hint: Proposition 2.4.19 may be useful.)

   (b) Show that the converse is false by giving a counterexample. (See Exercise 2.4.7.)

5. **Read Section 2.8, pages 71-73 of the textbook.**

6. **Exercise 6, page 73 of the textbook.** Let \(X\) be a topological space with an equivalence relation \(\sim\) such that \(X/\sim\) is a Hausdorff space. If \(X\) is locally connected, show that \(X/\sim\) is locally connected.

7. **Exercise 8, page 74 of the textbook.** If \(X\) is a pathwise connected space and \(\sim\) is an equivalence relation on \(X\) such that \(X/\sim\) is Hausdorff, show that \(X/\sim\) is pathwise connected.