Homework set 9 – Solutions

Math 4171 – Renato Feres

1. Read the remaining parts of Section 2.6, pages 65 and 66.

2. Exercise 5, page 67 of the textbook, modified. Let \( \{(X_i, \mathcal{T}_i) : i \in I\} \) be a collection of topological spaces, and let \( X = \prod_{i} X_i \) have the product topology.

   (a) Show that if \( X \) is separable, then \( X_i \) is separable for each \( i \in I \).

   (b) Suppose \( I \) is finite, so that \( X = X_1 \times \cdots \times X_n \). If each \( X_i \) is separable, show that \( X \) is separable.

   (c) Show that if \( I \) is countable and \( X_i \) is separable for each \( i \in I \), then \( X \) is separable.

Note 1: in the original statement, one is to prove that \( X \) is separable if and only if \( I \) is countable and each \( X_i \) is separable. But this does not seem to be true. See, for example, Product of separable spaces by K.A. Ross and A.H. Stone in The Mathematical Monthly, April 1964, Vol. 71, No. 4, pp. 398-403.

Note 2: You may take for granted the facts proved in the appendix to the textbook that the product of finitely many countable sets is countable, and the union of countably many countable sets is countable.

Solution.

(a) Suppose \( X \) is separable and let \( D \) be a countable and dense subset of \( X \). For any given \( i \in I \) and any given open and non-empty set \( G \subseteq X_i \), the open and non-empty subset \( \pi_i^{-1}(G) \subseteq X \) must intersect \( D \). Let \( x \) be in this intersection. Then \( \pi_i(x) \) lies in \( G \). This means that \( \pi_i(D) \) is dense in each \( X_i \) and countable, so \( X_i \) is separable.

(b) Let \( D_i \) be a countable and dense subset of \( X_i \) for \( i = 1, \ldots, n \) and write \( D = D_1 \times \cdots \times D_n \). I claim that \( D \) is countable and dense in \( X \). It is countable since a finite product of countable sets is countable. (This is shown by a finite induction and an argument similar to the one used in the proof of Proposition A.5.4, page 131.) Let us show that \( D \) is dense in \( X \). Let \( x \in X \) and \( G \) any neighborhood of \( x \). Let \( G_1 \times \cdots \times G_n \subseteq G \) be a neighborhood of \( x \). (Recall that such rectangles constitute a base for the finite product topology.) Let \( y_i \in D_i \cap G_i \) for each \( i \). Then \( y = (y_1, \ldots, y_n) \in D \cap G \). Since this is true for every \( x \) and every neighborhood \( G \) of \( x \), we conclude that \( D \) is dense in \( X \).

(c) Without loss of generality we may assume that \( I = \mathbb{N} \). I’ll represent the elements of \( X \) as sequences \( (x_1, x_2, \ldots) \) where \( x_i \in X_i \). Fix an \( a = (a_1, a_2, \ldots) \in X \) and let \( D_1, D_2, \ldots \) be countable dense sets in \( X_1, X_2, \ldots \), respectively. For each \( n \in \mathbb{N} \), define

\[
E_n = D_1 \times D_2 \times \cdots \times D_n \times \{a_{n+1}\} \times \{a_{n+2}\} \times \cdots \quad \text{and} \quad E = \bigcup_{n=1}^{\infty} E_n.
\]

We know that a finite product of countable sets is countable, and that the union of countably many countable sets is countable. (Proposition A.5.6, page 132.) So \( E \) is countable.
I claim that $E$ is dense in $X$. To see that this is the case, let $x \in X$ and $G$ a neighborhood of $x$. According to Lemma 2.6.9, there is an $n \in \mathbb{N}$ and $y \in G$ such that $y_i = a_i$ for all $i > n$. Moreover, by part (b) of this exercise, $D_1 \times \cdots \times D_n$ is dense in $X_1 \times \cdots \times X_n$ for all $n \in \mathbb{N}$. Let $U \subseteq G$ be a neighborhood of $y$ in $X$ taken from the base of the product topology. We may assume $U$ has the form

$$U = U_1 \times \cdots \times U_n \times X_{n+1} \times X_{n+2} \times \cdots$$

for some $n \in \mathbb{N}$ sufficiently large. Now choose $(u_1, \ldots, u_n) \in (D_1 \cap U_1) \times \cdots \times (D_n \cap U_n)$. Then

$$u = (u_1, \ldots, u_n, a_{n+1}, a_{n+2}, \ldots) \in E \cap G.$$ 

This shows that $E$ is dense in $X$.

\[\diamond\]

3. **Exercise 6, page 67 of the textbook.** Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces, and let $X = \prod_i X_i$ have the product topology. If, for each $i \in I$, $C_i$ is a component of $X_i$, is $C = \prod_{i \in I} C_i$ a component of $X$?

**Solution.** Let us prove that $C$ is indeed a component.

According to Theorem 2.6.10, $C$ must be connected. Thus we need to show that $C$ is maximal. Let $U$ be a connected subset of $X$ containing $C$. The goal is to prove that $U \subseteq C$ (so that $U = C$). If this is not the case, let $x \in U \setminus C$. Then we have $x_i \notin C_i$ for some $i$. But $\pi_i(U)$ is a connected set in $X_i$ (being the continuous image of a connected set) that properly contains $C_i$. But this is a contradiction because $C_i$ is a component in $X_i$. Therefore $U \subseteq C$, hence $C$ is a component.

\[\diamond\]

4. **Exercise 7, page 67 of the textbook, modified.** Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces, and let $X = \prod_i X_i$ have the product topology.

(a) Show that if $X$ is locally connected, then each $X_i$ is locally connected. (Hint: Proposition 2.4.19 may be useful.)

(b) Show that the converse is false by giving a counterexample. (See Exercise 2.4.7.)

**Solution.**

(a) Suppose that $X$ is locally connected. For each $i \in I$, we know that the coordinate projection $\pi_i$ is continuous, open and surjective. By Proposition 2.4.19, its image $X_i$ is locally connected.

(b) Given the statement of Exercise 2.4.7 (showing that the converse is true for a finite set $I$), we need to look for an example among infinite product spaces. Let us consider the simplest: let $X_I = \{0, 1\}$ with the discrete topology. This space is locally connected (since $\{0\}$ and $\{1\}$ are open and connected), although not connected. Let $X = \{0, 1\}^{\aleph_0}$. Suppose for a contradiction that $U$ is a connected neighborhood of $x = (0, 0, 0, \ldots)$. Now observe the following: base elements for the product topology are the intersections of finitely many sets of the form $\pi_i^{-1}(G_i)$ where $G_i$ is open in $X_i$. In other words, only finitely many coordinates are restricted in any way. This means that there are infinitely many $i \in I$ such that $\pi_i(U) = X_i$. Fix one such $i$. Then

$$U = \{U \cap \pi_i^{-1}(0)\} \cup \{U \cap \pi_i^{-1}(1)\}.$$ 

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But $A = U \cap \pi_i^{-1}(0))$ and $B = U \cap \pi_i^{-1}(1))$ are disjoint, open, non-empty subsets of $X$. Therefore $U$ could not be connected. Consequently, the product could not be locally connected.

\doit{5. Read Section 2.8, pages 71-73 of the textbook.}

\doit{6. Exercise 6, page 73 of the textbook.} Let $X$ be a topological space with an equivalence relation $\sim$ such that $X/\sim$ is a Hausdorff space. If $X$ is locally connected, show that $X/\sim$ is locally connected.

\textbf{Solution.} I’ll write $\overline{X}$ for $X/\sim$ and draw a line over the letters representing points and sets in $\overline{X}$. Let $\overline{x} \in \overline{X}$ and $\overline{G}$ a neighborhood of $\overline{x}$. Let $\overline{C}$ be the connected component of $\overline{G}$ containing $\overline{x}$. To show $\overline{X}$ is locally connected, it suffices to prove that $\overline{C}$ is open. And, by the definition of the quotient topology, $\overline{C}$ will be open if we can show $C := q^{-1}(\overline{C})$ is open.

We define $G = q^{-1}(\overline{G})$. Then $G$ is open (since $q$ is continuous) and contains $C$. Now let $x \in C$. Let $U$ be the connected component of $G$ that contains $x$. Note that $U$ is open since $X$ is assumed to be locally connected. (In fact, as $X$ is locally connected, any $y \in U$ has a connected neighborhood contained in $G$; but this connected neighborhood must be contained in $U$ by maximality of the component.)

Observe that $q(U)$ is connected and contains $\overline{x}$, so $q(U) \subseteq \overline{C}$ by the maximality of $\overline{C}$. Therefore $U \subseteq C$, hence $x$ is an interior point of $C$. Consequently, $C$ is open, which is what needed to be proved to conclude that $\overline{X}$ is locally connected.

Remark: I cannot see where the Hausdorff property is needed!

\doit{7. Exercise 8, page 74 of the textbook.} If $X$ is a pathwise connected space and $\sim$ is an equivalence relation on $X$ such that $X/\sim$ is Hausdorff, show that $X/\sim$ is pathwise connected.

\textbf{Solution.} Let $q(x), q(y)$ be any two elements of the quotient $X/\sim$, and let $f : [0, 1] \rightarrow X$ be a path in $X$ such that $f(0) = x$ and $f(1) = y$. Then $q \circ f$ defines a path from $q(x)$ to $q(y)$.

(Here again, I don’t see the need for the Hausdorff hypothesis.)