1. **Exercise 1, Section 3.2, page 84.** If $X$ is the real line with the topology generated by the subbase consisting of all the open intervals and the set $\mathbb{Q}$, show that $X$ is Hausdorff but not regular.

**Solution.** It is not hard to check that the given collection of subsets indeed forms a subbase for a topology on the real line since the open intervals themselves already form a subbase. This topology is clearly Hausdorff since any two distinct points can be separated by disjoint open intervals. But the topology is not regular. In fact, let $x$ be a rational number and $F = \mathbb{R} \setminus \mathbb{Q}$, the set of irrational numbers. In the given topology, $\mathbb{Q}$ is an open set, so its complement $F$ is closed. I claim that $x$ and $F$ cannot be separated by disjoint open sets.

To prove this claim, let $U$ be any neighborhood of $x$ and $V$ any neighborhood of $F$. We wish to show that $U \cap V$ cannot be empty. We may assume that $U$ is a base element. Note that base sets can be of two types: an open interval or the intersection of an open interval with $\mathbb{Q}$. As $x$ is rational, we may assume that $U = (a, b) \cap \mathbb{Q}$ for some open interval $(a, b)$ containing $x$. Let $y$ be any element of $F$ which is in $(a, b)$. (Such $y$ exists because the set $F$ of irrational numbers is dense in the real line.) Let $G$ be a base neighborhood of $y$ contained in $V$. Since $G$ contains the irrational point $y$, it must be an open interval: $G = (c, d)$ for real numbers $c < d$. Without loss of generality (as $a < y < b$), we may assume $a < c < d < b$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$, the interval $(c, d)$ must contain a rational point, which would then be a point in $U$. We conclude that $U$ and $V$ must intersect, so $X$ cannot be regular.

2. **Exercise 2, Section 3.2, page 84.** Prove that if $X$ is a regular topological space and $E \subseteq X$, then $E$ with its relative topology is regular.

**Solution.** Let $F$ be a closed subset of $E$ and $x$ a point of $E$ in the complement of $F$. A closed set in the subspace topology is the intersection $F' \cap E$ of a closed set $F'$ of $X$ with $E$. Clearly $F'$ cannot contain $x$ since $x$ belongs to $E \cap (X \setminus F)$. Since $X$ is regular, there must be disjoint open sets $U, V$ in $X$ such that $x \in U$ and $F' \subseteq V$. But then $U \cap E$ and $V \cap E$ are disjoint open sets in $E$ such that $x \in U \cap E$ and $F \subseteq V \cap E$. This shows that $E$ is also regular.

3. **Exercise 4, Section 3.2, page 84.** If $X$ is a topological space and for each $x \in X$ there is an open set $G$ such that $x \in G$ and $\text{cl}G$ with its relative topology is a regular space, then $X$ is regular.

**Solution.** Let $F$ be a closed subset of $X$ and $x \in X \setminus F$. Let $G$ be an open neighborhood of $x$ whose closure is regular with the relative topology. Let $F' = F \cap \text{cl}G$, which is a closed subset of $\text{cl}G$. Since the closure of $G$ is regular, we can find disjoint open subsets $U, W'$ of $\text{cl}G$ such that $x \in U$ and $F' \subseteq W'$. We may assume that $U$ is
an open subset of $G$ since $x \in G$ and $G$ is open. Let $W$ be an open subset of $X$ such that $W' = W \cap \text{cl}G$ and define $V = W \cup (X \setminus \text{cl}G)$. Then $U$ and $V$ are disjoint open sets in $X$, $x \in U$ and

$$F = (F \cap \text{cl}G) \cup (F \cap (X \setminus \text{cl}G)) \subseteq W \cup (X \setminus \text{cl}G) = V.$$ 

Therefore $X$ is regular.

\[\diamondsuit\]

5. **Exercise 5, Section 3.2, page 84, modified.** Show that if $X$ is regular then for any two distinct points $x$ and $y$ there are open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $\text{cl}U \cap \text{cl}V = \emptyset$. (The textbook claims that the converse is true. Can you prove it?)

**Solution.** Assume $X$ is regular. By the standing assumption that $X$ is Hausdorff, given two points $x, y$, we can find disjoint open sets $G$ and $H$ such that $x \in G$ and $y \in H$. Then by Proposition 3.2.2, there are open sets $U \subseteq \text{cl}U \subseteq G$ and $V \subseteq \text{cl}V \subseteq H$ such that $x \in U$, $y \in V$. But this means that the closures of $U$ and $V$ are disjoint.

\[\diamondsuit\]

6. **Exercise 10, page 84.** If $X$ is a connected completely regular space that is not a singleton, show that $X$ has uncountably many points.

**Solution.** Let two distinct points $x_1, x_2$ of $X$. Naturally, the set $\{x_1\}$ is closed (a consequence of the standing assumption that our topological spaces are Hausdorff). Since $X$ is completely regular, there is a continuous function $f : X \to \mathbb{R}$ such that $f(x_1) = 0$ and $f(x_2) = 1$. Since $X$ is connected, I claim that every $c \in [0, 1]$ is the value of some point of $X$ under $f$, and therefore $X$ has uncountably many points (since the unit interval is uncountable). In fact, suppose some $c \in (0, 1)$ is not the value of any point in $X$. Let $U = \{x \in X : f(x) > c\}$ and $V = \{x \in X : f(x) < c\}$. Then $U$ and $V$ are nonempty, disjoint, open subsets of $X$ whose union is all of $X$. But this contradicts the assumption that $X$ is connected.