Homework set 1 - due 09/16/22

Math 444

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The main reference for the following is Michel Le Bellac's *Quantum Physics*, Chapter 1 (Introduction). See also Brian C. Hall's *Quantum Theory for Mathematicians*, Chapter I.

1. Read Chapter 1 of Le Bellac's text.

2. In Planck's model of blackbody radiation, the energy in a given frequency ω of electromagnetic radiation is distributed randomly over all values $n\hbar\omega$, $n = 0, 1, 2, \dots$ so that the probability of energy $E = n\hbar\omega$ is, under the Boltzmann distribution, proportional to $\exp(-\beta n\hbar\omega)$.

   (a) Show that the partition function (this is the normalization factor) for the quantum harmonic oscillator is given by

   $$Z(\beta) := \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega} = \frac{1}{1 - \exp(-\beta\hbar\omega)}.$$

   Therefore,

   $$\mathrm{Prob}(\mathrm{Energy} = n\hbar\omega) = \frac{e^{-\beta n\hbar\omega}}{Z(\beta)} = e^{-\beta n\hbar\omega}\left[1 - \exp(-\beta\hbar\omega)\right].$$

   (b) The mean value for the energy is

   $$\langle E\rangle := \frac{1}{Z(\beta)} \sum_{n=0}^{\infty} n\hbar\omega e^{-\beta n\hbar\omega}.$$

   Show that

   $$\langle E\rangle = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}.$$

   For this purpose, first check the sum

   $$\sum_{n=0}^{\infty} n e^{-an} = \frac{e^{-a}}{(1 - e^{-a})^2}.$$

   (c) Check that $\langle E\rangle$ behaves like $1/\beta = k_b T$ for small values of ω, but decays exponentially as ω tends to infinity.

   (d) Suppose that the cavity for the blackbody radiation is a parallelepiped as in Figure 1.4, page 13, of Le Bellac's text. Thus the possible frequencies are (see Equation (1.13) in that text)

   $$\omega = 2\pi c\sqrt{\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2 + \left(\frac{n_3}{L_3}\right)^2} \tag{1}$$

   where the n_1, n_2, n_3 are integers and L_1, L_2, L_3 are the side lengths of the parallelepiped. Give a rough explanation for the following statement: For large values of the frequency ω, the number of the discrete frequencies given by Equation (1) in the range between ω_0 and $\omega_0 + \delta$, for δ small relative to ω_0, is proportional to

$\delta\omega_0^2$. Therefore, the amount of energy per unit of frequency is

$$f(\omega, T) = C\frac{\hbar\omega^3}{e^{\beta\hbar\omega} - 1},$$

where C is a constant involving the volume of the cavity and the speed of light c. This is *Planck's law*. The relationship between the shape of the cavity and the number of frequencies is known as *Weyl's law*.

3. Going back the Maxwell's equations (1.8) and (1.9) in Le Bellac's text (page 10), and using basic vector calculus identities, show that the electric and magnetic fields **E** and **B** satisfy the wave equation with speed coefficient c as given in (1.10).

4. In an interference experiment using fullerenes C_{60}, which are at present the largest objects for which wave behavior has been verified experimentally, the average speed of the molecules is about $220ms^{-1}$ What is their de Broglie wavelength? How does it compare with the size of the molecule? (Some web research may be needed.)