There are two parts to this assignment. In the first part we look at a class of problems in quantum probability theory that requires the use of symmetric and antisymmetric tensor products of Hilbert spaces. In physics, this is related to the subject of quantum statistics and the notions of fermions and bosons. For more on this topic (from a math perspective) see An Introduction to Quantum Stochastic Calculus by K.R. Parthasarathy (Springer, 1992, Chapter 2, Section 17) and Indistinguishable Classical Particles by Alexander Bach (Springer, 1997).

The second part has to do with the postulate about the dynamics of quantum states.

In the first part, most of the exercises, except number 6, already have answers. You should take those as a reading assignment.

量子统计学。考虑一个由n个相同子系统的系统。我们将这些子系统称为粒子。说这些粒子是相同的意味着它们在所有内在（也就是说，状态不依赖）特性上一致。实际上，对于这个作业，这意味着相同的粒子与一个相同的希尔伯特空间相关联。如果H表示一个子系统的希尔伯特空间，那么复合系统的希尔伯特空间是n-次张量积H⊗n.

相同的粒子被称为不可区别的，如果它们处于一个状态，这个状态是对称的。要理解这个声明，回想S_n是n符号的对称群，也就是说，所有对称的组态。对于所有σ ∈ S_n，定义π(σ)为线性映射

$$\pi(\sigma) u_1 \otimes \cdots \otimes u_n = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(n)}$$

对可分解的张量，然后通过线性性将π(σ)扩展到所有H⊗n的映射。回忆π是一个从S_n到H⊗n的对称群的群同态。这意味着π(σ)是一个单元算符，并且π(ση) = π(σ)π(η)对于所有σ, η ∈ S_n。

我们可以现在将状态定义为相同的粒子是不可区分的，如果它们在一种状态（由密度算符ρ定义）中是相同的，这种状态在π(σ)ρπ(σ^{-1}) = ρ对所有σ ∈ S_n。例如，如果ρ_0是一个状态上的H算符，则ρ_0⊗n := ρ_0⊗⋯⊗ρ_0是一个对称状态。

1. 让ρ是一个对称状态的一个复合系统的n个相同的粒子。每个粒子的Hilbert空间是H。让P_1,...,P_n ∈ P(H)是H的正交投影算符，代表事件，且让联合事件为P := P_1⊗⋯⊗P_n。定义π(σ)将联合事件π(σ)Pπ(σ^{-1})。 (这是线性映射的合成)

   (a) 显示

$$\pi(\sigma) P \pi(\sigma^{-1}) = P_{\sigma^{-1}(1)} \otimes \cdots \otimes P_{\sigma^{-1}(n)}.$$
(b) Show that the probability of any permuted event $\Pi(\sigma)P\Pi(\sigma^{-1})$ with respect to the symmetric state $\rho$ is the same as the probability of $P$ with respect to $\rho$. (Recall that the probability of the event $P$ in the state $\rho$ is $\text{Tr}(\rho P)$.)

(c) Let $P_s$ and $P_a$ be the linear transformations on $HH^n$ defined by

$$P_s := \frac{1}{n!} \sum_{\sigma \in S_n} \Pi(\sigma), \quad P_a := \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma)\Pi(\sigma),$$

called the symmetrization and antisymmetrization operators. Show that $P_s, P_a \in \mathcal{P}(HH^n)$. (Recall that the latter space consists of the orthogonal projections on the tensor product Hilbert space. Thus proving that $P \in \mathcal{P}(HH^n)$ amounts to checking that $P^2 = P$ and that $P$ is self-adjoint with respect to the inner product on $HH^n$.)

(d) Show that

$$\Pi(\sigma)P_s = P_s\Pi(\sigma) = P_s, \quad \Pi(\sigma)P_a = P_a\Pi(\sigma) = \text{sgn}(\sigma)P_a$$

for all $\sigma \in S_n$.

(e) Let $HH^n_s$ be the range of $P_s$ (i.e., the image of $HH^n$ under the transformation $P_s$). We call it the symmetrized tensor product. Similarly, define the antisymmetrized tensor product $HH^n_a$, the range of $P_a$. Show that the symmetric state $\rho$ maps the symmetrized tensor product into itself and the antisymmetrized tensor product into itself.

(f) Let $\rho$ be a symmetric state of $n$ identical particles. Let $u \in HH^n$, $u \neq 0$, be an eigenvector of $\rho$ corresponding to eigenvalue $p$. Show that the linear span of all $\Pi(\sigma)u$, $\sigma \in S_n$, is contained in the eigenspace of $\rho$ for the eigenvalue $p$. (This amounts to checking that $\rho\Pi(\sigma)u = p\Pi(\sigma)u$ for all $\sigma \in S_n$.)

We say that the $n$ particles are $n$ identical bosons if their joint Hilbert space is $HH^n_s$ with a symmetric state. They are $n$ identical fermions if their joint Hilbert space is $HH^n_a$ with a symmetric state.

**Solution.**

(a) Let us apply $\Pi(\sigma)P\Pi(\sigma^{-1})$ to a decomposable vector $u_1 \otimes \cdots \otimes u_n$:

$$\Pi(\sigma)P\Pi(\sigma^{-1})u_1 \otimes \cdots \otimes u_n = \Pi(\sigma)Pu_{\sigma(1)} \otimes \cdots \otimes Pu_{\sigma(n)}$$

$$= \Pi(\sigma)P_1u_{\sigma(1)} \otimes \cdots \otimes P_nu_{\sigma(n)}$$

$$= P_{\sigma^{-1}(1)}u_1 \otimes \cdots \otimes P_{\sigma^{-1}(n)}u_n$$

$$= P_{\sigma^{-1}(1)} \otimes \cdots \otimes P_{\sigma^{-1}(n)}u_1 \otimes \cdots \otimes u_n.$$  

As decomposable tensors of the form $u_1 \otimes \cdots \otimes u_n$ span $HH^n$, the claimed identity holds.

(b) If $\rho$ is a symmetric state and $P$ then, since $\text{Tr}(AB) = \text{Tr}(BA)$,

$$\text{Tr}(\rho\Pi(\sigma)P\Pi(\sigma^{-1})) = \text{Tr}(\Pi(\sigma^{-1})\rho\Pi(\sigma)P) = \text{Tr}(\rho P).$$

(c) I will show this for $P_a$. The same argument holds for $P_s$. First note that $\Pi(\sigma)$ satisfies

$$\Pi(\sigma)^* = \Pi(\sigma^{-1}).$$
It suffices to verify this identity on separable tensors:

\[
\langle \Pi(\sigma)^* u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle = \langle u_1 \otimes \cdots \otimes u_n, \Pi(\sigma) v_1 \otimes \cdots \otimes v_n \rangle \\
= \langle u_1 \otimes \cdots \otimes u_n, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \rangle \\
= \langle u_1, v_{\sigma^{-1}(1)} \rangle \cdots \langle u_n, v_{\sigma^{-1}(n)} \rangle \\
= \langle u_{\sigma(1)}, v_1 \rangle \cdots \langle u_{\sigma(n)}, v_n \rangle \quad \text{(permuting the factors of this product according to } \sigma) \\
= \langle u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}, v_1 \otimes \cdots \otimes v_n \rangle \\
= \langle \Pi(\sigma^{-1}) u_1 \otimes \cdots \otimes u_t, v_1 \otimes \cdots \otimes v_n \rangle.
\]

Therefore, since \(\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})\),

\[
P_a^* = \frac{1}{n!} \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Pi(\sigma) \right)^* = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Pi(\sigma)^* = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Pi(\sigma^{-1}) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma^{-1}) \Pi(\sigma^{-1}).
\]

The group inverse \(\sigma \rightarrow \sigma^{-1}\) is a bijection. Therefore

\[
\frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma^{-1}) \Pi(\sigma^{-1}) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Pi(\sigma) = P_a.
\]

This shows that \(P_a^* = P_a\). Let us now check that \(P_a^2 = P_a\).

\[
P_a P_a = \left( \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Pi(\sigma) \right) \left( \frac{1}{n!} \sum_{\eta \in S_n} \text{sgn}(\eta) \Pi(\eta) \right) \\
= \frac{1}{n!} \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\eta \in S_n} \text{sgn}(\sigma) \text{sgn}(\eta) \Pi(\sigma) \Pi(\eta) \\
= \frac{1}{n!} \sum_{\sigma \in S_n} \left( \frac{1}{n!} \sum_{\eta \in S_n} \text{sgn}(\sigma \eta) \Pi(\sigma \eta) \right) \quad \text{(sgn and } \Pi \text{ are homomorphisms)} \\
= \frac{1}{n!} \sum_{\sigma \in S_n} P_a \quad (\eta \rightarrow \sigma \eta \text{ is a bijection}) \\
= P_a \quad (S_n \text{ has order } n!)
\]

(d) Let us check this for \(P_a\). The same argument will apply to \(P_s\). Since

\[
\Pi(\sigma) \Pi(\eta) = \Pi(\sigma) \Pi(\eta), \quad \text{sgn}(\sigma \eta) = \text{sgn}(\sigma) \text{sgn}(\eta), \quad \text{sgn}(\sigma) = \text{sgn}(\sigma^{-1}),
\]

then

\[
\Pi(\sigma) P_a = \frac{1}{n!} \sum_{\eta \in S_n} \text{sgn}(\eta) \Pi(\sigma \eta) = \frac{1}{n!} \sum_{\eta \in S_n} \text{sgn}(\sigma^{-1}) \text{sgn}(\sigma \eta) \Pi(\sigma \eta) = \text{sgn}(\sigma) \frac{1}{n!} \sum_{\eta \in S_n} \text{sgn}(\eta) \Pi(\eta) = \text{sgn}(\sigma) P_a.
\]

We have used that \(\eta \rightarrow \sigma \eta\) is a bijection. The other identities are shown in a similar way.

(e) It suffices to check that \(\rho\) sends elements of \(\mathcal{H}_a^{\otimes n}\) of the form \(P_a u, u \in \mathcal{H}^{\otimes n}\), to elements of the same form.

But \(\rho\) commutes with each \(\Pi(\sigma)\), therefore it commutes with \(P_a\). Thus \(\rho P_a u = P_a \rho u\). The same argument applies to \(P_s\).

(f) This is immediate due to \(\rho\) commuting with \(\Pi(\sigma)\).
2. Let \([e_1, \ldots, e_N]\) be an orthonormal basis for the \(N\)-dimensional Hilbert space \(\mathcal{H}\). Let \(r_1, \ldots, r_N\) be nonnegative integers such that \(r_1 + \cdots + r_N = n\). It will be useful to employ the following notation:

\[
e(r_1, \ldots, r_N) := \sum_{r_1}^{e_1} \otimes \cdots \otimes \sum_{r_i}^{e_i} \otimes \cdots \otimes \sum_{r_N}^{e_N}.
\]

Naturally, if \(r_i = 0\), the term \(e_i\) does not appear in the above product. The orthogonal projection to the one-dimensional subspace of \(\mathcal{H}^{\otimes n}\) spanned by \(e(r_1, \ldots, r_N)\) will be denoted \(|e(r_1, \ldots, r_N))\rangle \langle e(r_1, \ldots, r_N)|\).

(a) Show that \(\left\{ \left( \frac{n!}{r_1! \cdots r_N!} \right)^{1/2} P_s e(r_1, \ldots, r_N) : r_j \geq 0 (j = 1, \ldots, N), r_1 + \cdots + r_N = n \right\}\)

is an orthonormal basis for \(\mathcal{H}_s^{\otimes n}\). Conclude that

\[
\dim \mathcal{H}_s^{\otimes n} = \binom{N + n - 1}{n}.
\]

(b) Show that \(\mathcal{H}_a^{\otimes n} = \{0\}\) if \(n > N\). If \(n \leq N\), show that

\[
\left\{ \left( \frac{n!}{r_1! \cdots r_N!} \right)^{1/2} P_s e(r_1, \ldots, r_N) : r_j \in \{0, 1\}, r_1 + \cdots + r_N = n \right\}
\]

is an orthonormal basis for \(\mathcal{H}_a^{\otimes n}\). Conclude that

\[
\dim \mathcal{H}_a^{\otimes n} = \binom{N}{n}.
\]

Solution.

(a) Let us first check that these vectors are orthonormal. Let us define

\[
a_{rs} := \left( \frac{n!}{r_1! \cdots r_N!} \right)^{1/2} P_s e(r_1, \ldots, r_N), \left( \frac{n!}{s_1! \cdots s_N!} \right)^{1/2} P_s e(s_1, \ldots, s_N) \right).\]

Then, since \(P_s^2 = P_s = P_s^2\),

\[
a_{rs} = \frac{n!}{\sqrt{r_1! \cdots r_N! s_1! \cdots s_N!}} \langle e(r_1, \ldots, r_N), P_s e(s_1, \ldots, s_N) \rangle
\]

\[
= \frac{1}{\sqrt{r_1! \cdots r_N! s_1! \cdots s_N!}} \sum_{\sigma \in S_n} \langle e(r_1, \ldots, r_N), \Pi(\sigma) e(s_1, \ldots, s_N) \rangle
\]

\[
= \frac{1}{\sqrt{r_1! \cdots r_N! s_1! \cdots s_N!}} \sum_{\sigma \in S_n} \left( e_{r_1} \otimes \cdots \otimes e_{r_i} \otimes \cdots \otimes e_{r_N} \otimes e_{s_1} \otimes \cdots \otimes e_{s_i} \otimes \cdots \otimes e_{s_N} \right)^{1/2} P_s e(s_1, \ldots, s_N) \right).
\]

It is apparent from this last expression that it is nonzero only when \((r_1, \ldots, r_N) = (s_1, \ldots, s_N)\) and when \(\sigma\) permutes the factors in each block \(e_i \otimes \cdots \otimes e_i\) among themselves. The number of permutations of this kind is \(r_1! \cdots r_N!\) and each nonzero inner product in the above sum equals 1. Therefore \(a_{rs} = 0\) if \(r \neq s\) and \(a_{rr} = 1\). Further note that the symmetrization of an arbitrary tensor of the form \(e_{r_1} \otimes \cdots \otimes e_{r_n}\) equals
$P_s e(r_1, \ldots, r_N)$ for some $(r_1, \ldots, r_N)$ such that $r_1 + \cdots + r_N = n$. Therefore the orthonormal family of tensors spans all of $\mathcal{H}_s^\otimes n$ and thus constitutes an orthonormal basis.

To show that the dimension is the given binomial expression, we need to count the ways we can separate the numbers $1, 2, \ldots, n$ into $N$ bunches of consecutive numbers. We think of the bunches as obtained by placing $N-1$ walls between the numbers disposed in linear fashion. Thus let us imagine that there are a total of $n+N-1$ places disposed in a line and that each place is to be occupied by either a number from $1$ to $n$ (in this order) or by a wall. The number of bunches is then the number of ways of choosing $n-1$ places out of $n+N-1$. This is also equal to the number of ways of choosing $n$ out of $N+n-1$, which is the given binomial expression.

(b) First note that $P_a e(r_1, \ldots, r_N) = 0$ if $r_i \geq 2$ for some $i$. In fact, let $\sigma$ be an odd permutation that only permutes the factors in the block $e_i \otimes \cdots \otimes e_i$. Then

$$-P_a e(r_1, \ldots, r_N) = \Pi(\sigma) P_a e(r_1, \ldots, e_N) = P_a \Pi(\sigma) e(r_1, \ldots, r_N) = P_a e(r_1, \ldots, r_N).$$

This implies that $P_a e(r_1, \ldots, r_N) = 0$. Therefore the image of $P_a$ is spanned by the vectors

$$\{P_a e(r_1, \ldots, r_N), r_i \in [0, 1], r_1 + \cdots + r_N = n\}.$$

We can show that the vectors $\sqrt{m} P_a e(r_1, \ldots, r_N)$ are orthonormal by the same argument used in the previous item. Thus for the antisymmetrized tensor product Hilbert space to be non-zero we must have $N \geq n$.

If we write these antisymmetrized vectors as $P_a e_{i_1} \otimes \cdots \otimes e_{i_n}, i_1 < \cdots < i_n$, then the number of such vectors equals the number of ways of choosing $n$ (ordered) indices among $N$ numbers. But this is the given binomial coefficient.

\diamond

3. Statistics of occupancy I: Maxwell-Boltzmann. Let us first consider a classical probability problem. Suppose $n$ identical balls are placed randomly into $N$ urns. The urns are labeled $1, \ldots, N$. The probability that any one of the balls will fall into urn $j$ is assumed to be $p_j \geq 0$, so that $p_1 + \cdots + p_N = 1$. We wish to determine the probability of the event $E(r_1, \ldots, r_N)$ of obtaining $r_j$ balls in urn $j$ for $j = 1, \ldots, N$. It is a basic fact from probability theory that the probability of this event is

$$\Pr(E(r_1, \ldots, r_N)) = \frac{n!}{r_1! \cdots r_N!} p_1^{r_1} \cdots p_N^{r_N}.$$

This is called the multinomial distribution. Let us derive this probability using the language of Hilbert spaces and linear operators.

The probability distribution for a single ball, written as a density operator, is

$$\rho = \sum_{j=1}^N p_j |j\rangle \langle j|.$$

Here, for a single ball, we employ the standard notation $|j\rangle = e_j$. Note that the probability of finding a given ball with density operator $\rho$ in urn $s$ is the probability of the event $E_s := |s\rangle \langle s| \in \mathcal{B}(\mathcal{H})$, which is

$$\Pr_{\rho_0}(E_s) = \Tr(\rho_0 E_s) = \Tr\left( \sum_j p_j |j\rangle \langle j| s\rangle \langle s| \right) = \sum_j p_j |\langle j| s\rangle|^2 = \sum_j p_j \delta_{js} = p_s.$$
We assume that the joint probability distribution for all the balls is given by \( \rho = \rho_0^{\otimes n} = \rho_0 \otimes \cdots \otimes \rho_0 \). To verify that the multinomial distribution indeed holds, we need to obtain the probability of the event \( E(r_1, \ldots, r_N) \), which may be written in Dirac notation as

\[
E(r_1, \ldots, r_N) = \sum_{|i: j_i = j| = r_j} \langle j_1 \cdots j_n | j_1 \cdots j_n \rangle,
\]

where the summation is over all \( (j_1, \ldots, j_n) \in \{1, \ldots, N\}^n \) such that each value \( j \in \{1, \ldots, N\} \) appears \( r_j \) times.

(a) Show that \( E(r_1, \ldots, r_N) \) is an orthogonal projection on \( \mathcal{H}^{\otimes n} \) whose range has dimension \( \frac{n!}{r_1! \cdots r_N!} \).

(b) Show that the probability of the event \( E(r_1, \ldots, r_N) \) in state \( \rho_0^{\otimes n} \) is given by

\[
\text{Tr}(\rho_0^{\otimes n} E(r_1, \ldots, r_N)) = \frac{n!}{r_1! \cdots r_N!} p_1^{r_1} \cdots p_N^{r_N}.
\]

### Solution.

(a) That \( E(r_1, \ldots, r_N) \) is an orthogonal projection is an immediate consequence of the fact that it is a sum of orthogonal projections that are mutually orthogonal. The dimension of the range is the dimension of the subspace of \( \mathcal{H}^{\otimes n} \) spanned by the orthogonal unit vectors \( |j_1 \cdots j_n\rangle \) such that each value \( j \in \{1, \ldots, N\} \) appears \( r_j \) times among the \( j_i \). But the number of such vectors is the number of ways of distributing \( n \) (distinct) balls into \( N \) urns such that urn \( j \) has \( r_j \) balls. A standard counting method shows that this number is given by the multinomial coefficient

\[
\binom{n}{r_1, \ldots, r_N} = \frac{n!}{r_1! \cdots r_N!}.
\]

(b) We have

\[
\text{Tr}(\rho_0^{\otimes n} E(r_1, \ldots, r_N)) = \sum_{|i: j_i = j| = r_j} \text{Tr}(\rho_0^{\otimes n} |j_1 \cdots j_n\rangle \langle j_1 \cdots j_n|)
\]

\[
= \sum_{|i: j_i = j| = r_j} \langle j_1 \cdots j_n | \rho_0^{\otimes n} | j_1 \cdots j_n \rangle
\]

\[
= \sum_{|i: j_i = j| = r_j} p_1 \cdots p_N
\]

\[
= \frac{n!}{r_1! \cdots r_N!} p_1^{r_1} \cdots p_N^{r_N}.
\]

\( \diamond \)

4. **Statistics of occupancy II: Bose-Einstein.** We suppose that the \( n \) balls are \( n \) identical bosons. The Hilbert space for the system is now \( \mathcal{H}^{\otimes n}_s \). We take the probability distribution \( \rho_0^{\otimes n} \) conditional on the event \( P_s \). (Recall that \( P_s \) is the orthogonal projection to the symmetrized tensor product Hilbert space.) It makes sense to define the conditional probability distribution by the normalized density operator

\[
\rho_s^{\otimes n} := \frac{\rho_0^{\otimes n} | \mathcal{H}^{\otimes n}_s \rangle \langle \mathcal{H}^{\otimes n}_s |}{\text{Tr}(\rho_0^{\otimes n} P_s)}.
\]
(It should be clear that $\rho_0^{\otimes n}$ maps the symmetrized Hilbert space into itself.) Thus a system of $n$ indistinguishable bosons may be defined by the quantum probability space

$$\left(\mathcal{H}_s^{\otimes n}, \mathcal{P}\left(\mathcal{H}_s^{\otimes n}\right), \rho_0^{\otimes n}\right).$$

We now wish to obtain the probability of the event that the $n$ bosons are distributed over the $N$ urns according to the occupation numbers $(r_1, \ldots, r_N)$. Imposing the symmetry on the event $E(r_1, \ldots, r_N)$, we have

$$E_s(r_1, \ldots, r_N) := P_s E(r_1, \ldots, r_N) P_s.$$

Note that this event corresponds to the orthogonal projection to the one-dimensional subspace of $\mathcal{H}_s^{\otimes n}$ spanned by the vector $P_s | e(r_1, \ldots, r_N) \rangle$.

(a) Show that $E_s(r_1, \ldots, r_N)$ is the orthogonal projection to the one-dimensional subspace of $\mathcal{H}_s^{\otimes n}$ spanned by the tensor $\left(\frac{n!}{r_1! \cdots r_N!}\right)^{1/2} P_s | e(r_1, \ldots, r_N) \rangle$.

(b) Show that

$$c := \text{Tr} \left( \rho_0^{\otimes n} P_s \right) = \sum_{s_1 + \cdots + s_N = n} p_1^{s_1} \cdots p_N^{s_N}.$$

(c) Show that the probability of the event $E_s(r_1, \ldots, r_N)$ in the bosonic state $\rho_0^{\otimes n} = c^{-1} \rho_0^{\otimes n} | \mathcal{H}_s^{\otimes n} \rangle$ is

$$\text{Tr} \left( \rho_0^{\otimes n} E_s(r_1, \ldots, r_N) \right) = \frac{p_1^{s_1} \cdots p_N^{s_N}}{\sum_{s_1 + \cdots + s_N = n} p_1^{s_1} \cdots p_N^{s_N}}.$$

In this case, the indistinguishable particles are said to obey the Bose-Einstein statistics.

Solution.

(a) This follows from the observation that

$$P_s E(r_1, \ldots, r_N) P_s = \sum_{||i:j||=j} P_s | j_1 \cdots j_n \rangle \langle j_1 \cdots j_n | P_s$$

$$= \sum_{||i:j||=j} P_s | e(r_1, \ldots, r_N) \rangle \langle e(r_1, \ldots, r_N) | P_s$$

$$= \frac{n!}{r_1! \cdots r_N!} P_s | e(r_1, \ldots, r_N) \rangle \langle e(r_1, \ldots, r_N) | P_s.$$

which is the orthogonal projection to the one-dimensional subspace generated by the unit vector

$$\sqrt{\frac{n!}{r_1! \cdots r_N!}} P_s | e(r_1, \ldots, r_N) \rangle.$$
(b) We have, using the result of the first item:

\[
\text{Tr}(\rho_0^{\otimes n} P_s) = \sum_{s_1 + \cdots + s_N = n} \text{Tr}(\rho_0^{\otimes n} P_s e(s_1, \ldots, s_N) P_s)
\]

\[
= \sum_{s_1 + \cdots + s_N = n} \frac{n!}{s_1! \cdots s_N!} \text{Tr}(\rho_0^{\otimes n} P_s |e(s_1, \ldots, s_N)\rangle \langle e(s_1, \ldots, s_N)| P_s)
\]

\[
= \sum_{s_1 + \cdots + s_N = n} \frac{n!}{s_1! \cdots s_N!} \langle e(s_1, \ldots, s_N)| \rho_0^{\otimes n} P_s |e(s_1, \ldots, s_N)\rangle
\]

\[
= \sum_{s_1 + \cdots + s_N = n} \frac{n!}{s_1! \cdots s_N!} p_{s_1}^{s_1} \cdots p_{s_N}^{s_N} \langle e(s_1, \ldots, s_N)| P_s |e(s_1, \ldots, s_N)\rangle
\]

\[
= \sum_{s_1 + \cdots + s_N = n} \frac{n!}{s_1! \cdots s_N!} p_{s_1}^{s_1} \cdots p_{s_N}^{s_N} \sum_{\sigma \in S_n} \langle e(s_1, \ldots, s_N)| \Pi(\sigma)| e(s_1, \ldots, s_N)\rangle
\]

\[
= \sum_{s_1 + \cdots + s_N = n} \frac{n!}{s_1! \cdots s_N!} p_{s_1}^{s_1} \cdots p_{s_N}^{s_N} s_1! \cdots s_N!
\]

\[
= \sum_{s_1 + \cdots + s_N = n} p_{s_1}^{s_1} \cdots p_{s_N}^{s_N}.
\]

(c) The calculation of the previous item already shows that

\[
\text{Tr}(\rho_0^{\otimes n} E_s(r_1, \ldots, r_N)) = p_{r_1}^{s_1} \cdots p_{r_N}^{s_N}.
\]

Therefore,

\[
\text{Tr}(\rho_s^{\otimes n} E_s(r_1, \ldots, r_N)) = \frac{p_{r_1}^{s_1} \cdots p_{r_N}^{s_N}}{\sum_{s_1 + \cdots + s_N = n} p_{s_1}^{s_1} \cdots p_{s_N}^{s_N}}
\]

as claimed.

\[\diamondsuit\]

5. **Statistics of occupancy III: Fermi-Dirac.** Let the Hilbert space now be the antisymmetrized tensor product \( \mathcal{H}_a^{\otimes n} \). We define the state \( \rho_a^{\otimes n} \) as

\[
\rho_{a}^{\otimes n} := \frac{\rho_{0}^{\otimes n} | H_a^{\otimes n} \rangle \langle H_a^{\otimes n}|}{\text{Tr}(\rho_{0}^{\otimes n} E_a)}.
\]

A system of \( n \) indistinguishable fermions may be defined by the quantum probability space

\[
\left( \mathcal{H}_a^{\otimes n}, p \left( \mathcal{H}_a^{\otimes n}, \rho_{a}^{\otimes n} \right) \right).
\]

We now wish to obtain the probability of the event that the \( n \) fermions are distributed over the \( N \) urns according to the occupation numbers \((r_1, \ldots, r_N)\). Imposing the symmetry on the event \( E(r_1, \ldots, r_N) \), we have

\[
E_a(r_1, \ldots, r_N) := P_a E(r_1, \ldots, r_N) P_a.
\]

(a) Show that \( E_a(r_1, \ldots, r_N) \) is the orthogonal projection to the one-dimensional subspace of \( \mathcal{H}_a^{\otimes n} \) spanned by the vector \( \sqrt{n!} P_a | e(r_1, \ldots, r_N) \rangle \) where \( r_j \in \{0, 1\}, r_1 + \cdots + r_N = n \). Alternatively, we may write this vector as

\[
\sqrt{n!} P_a | j_1 \ldots j_n \rangle, \quad j_1 < j_2 < \cdots < j_n.
\]
(b) Show that
\[ c := \text{Tr}(\rho_0^n P_a) = \sum_{1 \leq j_1 < \cdots < j_n \leq N} p_{j_1} \cdots p_{j_n}. \]

(c) Show that the probability of the event \( E_a(r_1, \ldots, r_N) \) in the fermionic state \( \rho_a^n = c^{-1} \rho_0^n |P_a\rangle \langle P_a| \) is
\[ \text{Tr}(\rho_a^n E_a(r_1, \ldots, r_N)) = \frac{p_{j_1} \cdots p_{j_n}}{\sum_{1 \leq i_1 < \cdots < i_n \leq N} p_{i_1} \cdots p_{i_n}}. \]

In this case, the indistinguishable particles are said to obey the Fermi-Dirac statistics.

**Solution.**

(a) We have
\[ E_a(r_1, \ldots, r_N) = P_a E(r_1, \ldots, r_N) P_a \]
\[ = \sum_{|\langle i ; j \rangle| r_j = 1, \ldots, N} P_a |j_1 \cdots j_n \rangle \langle j_1 \cdots j_n| P_a \]
\[ = \sum_{|\langle i ; j \rangle| r_j = 1, \ldots, N} P_a |e(r_1, \ldots, r_N) \rangle \langle e(r_1, \ldots, e_N)| P_a. \]

Note that \( P_a e(r_1, \ldots, r_N) = 0 \) if \( r_j \geq 2 \) for some \( j \). In fact, if \( e_1 \otimes \cdots \otimes e_t \) is a block in \( e(r_1, \ldots, r_N) \) with at least \( 2 \) factors and \( \sigma \in S_n \) is a transposition which exchanges two equal factors in this tensor product, then \( \Pi(\sigma)e(r_1, \ldots, r_N) = e(r_1, \ldots, r_N) \). Since also \( \Pi(\sigma)P_a = \text{sgn}(\sigma)P_a \), we have
\[ P_a e(r_1, \ldots, r_N) = P_a \Pi(\sigma)e(r_1, \ldots, r_N) = \Pi(\sigma)P_a e(r_1, \ldots, r_N) = -P_a e(r_1, \ldots, r_N). \]

Therefore
\[ E_a(r_1, \ldots, r_N) = \sum_{|\langle i ; j \rangle| r_j = 0, 1; j = 1, \ldots, N} P_a |e(r_1, \ldots, r_N) \rangle \langle e(r_1, \ldots, e_N)| P_a. \]

The number of terms in the above summation is the number of \( n \)-tuples \( (j_1, \ldots, j_n) \) consisting of distinct \( j_i \), one for each urn that are occupied (corresponding to a \( r_j = 1 \)). This is the number of all permutations of the entries of \( (j_1, \ldots, j_n) \), which is \( n! \). Therefore,
\[ E_a(r_1, \ldots, r_N) = n! P_a |e(r_1, \ldots, r_N)\rangle \langle e(r_1, \ldots, e_N)| P_a = \left( \sqrt{n!} P_a |e(r_1, \ldots, r_N)\rangle \right) \left( \sqrt{n!} \langle e(r_1, \ldots, r_N)| \right) P_a. \]

(b) We have
\[ \text{Tr}(\rho_0^n E_a(r_1, \ldots, r_N)) = n! \text{Tr}(\rho_0^n P_a |j_1 \cdots j_n \rangle \langle j_1 \cdots j_n| P_a) \]
\[ = n! |j_1 \cdots j_n \rangle \langle j_1 \cdots j_n| P_a \rho_0^{\otimes n} |j_1 \cdots j_n \rangle \langle j_1 \cdots j_n| \]
\[ = p_{j_1} \cdots p_{j_n} n! |j_1 \cdots j_n \rangle \langle j_1 \cdots j_n| P_a |j_1 \cdots j_n \rangle \]
\[ = p_{j_1} \cdots p_{j_n}. \]

The probability of \( P_a \) in state \( \rho_0^{\otimes n} \) is the sum of the above probabilities over all the \( i_1 < \cdots < i_n \):
\[ \text{Tr}(\rho_0^{\otimes n} P_a) = \sum_{1 \leq i_1 < \cdots < i_n \leq N} p_{i_1} \cdots p_{i_n}. \]

Thus we conclude that
\[ \text{Tr}(\rho_0^{\otimes n} E_a(r_1, \ldots, r_N)) = \frac{p_{j_1} \cdots p_{j_n}}{\sum_{1 \leq i_1 < \cdots < i_n \leq N} p_{i_1} \cdots p_{i_n}}. \]
6. In order to compare the above three distributions, consider the case of \( n \) particles and 2 urns. Let the single particle state be

\[
\rho_0 = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1|.
\]

Show the following:

(a) According to the Maxwell-Boltzmann statistics the number of particles in urn 1 has a binomial distribution given by

\[
\text{Pr (urn 1 has } k \text{ particles)} = \binom{n}{k} 2^{-n}, \quad 0 \leq k \leq n.
\]

In particular, the probability that all the balls occupy the first urn is \( 2^{-n} \).

(b) According to the Bose-Einstein statistics

\[
\text{Pr (urn 1 has } k \text{ particles)} = \frac{1}{n+1}, \quad 0 \leq k \leq n.
\]

In particular, the probability that all the balls occupy the first urn is \( 1/(n+1) \).

(c) According to the Fermi-Dirac statistics it is not possible to have more than two particles when there are only two urns. Two particles cannot occupy the same urn. This is the so-called Pauli exclusion principle.

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**Time evolution of quantum states and the Schrödinger equation.** Let us begin with a few remarks about the dynamics of states in the classical setting. Let \( X \) be a set whose elements represent the classical (pure) states of the system. Here, we mean by a *state* something more than we meant previously. We suppose that the system can evolve in time in a deterministic way and that the state contains all the information we need in principle to uniquely determine the system’s future (and past) at any given time. Thus there exists a map \( \varphi \) such that, if \( x_0 \in X \) is the state of the system at time \( t_0 \), then at the time \( t_0 + \Delta t \) the state will be \( \varphi_{\Delta t}(x_0, t_0) \). At the time \( t_0 + \Delta t + \Delta t' \), the system will evolve from its state at time \( t_0 + \Delta t \) to the new state \( \varphi_{\Delta t'}(\varphi_{\Delta t}(x_0, t_0), t_0 + \Delta t) \). By uniqueness of the time evolution (determinism), this is the same state we obtain by starting from \( x_0 \) at \( t_0 \) and letting the system evolve over the time interval \( \Delta t + \Delta t' \). Thus

\[
\varphi_{\Delta t + \Delta t'}(x_0, t_0) = \varphi_{\Delta t'}(\varphi_{\Delta t}(x_0, t_0), t_0 + \Delta t).
\]

For simplicity, we will only consider the case in which the law governing the evolution of the system does not change in time (time homogeneity). This means that

\[
\varphi_{\Delta t}(x_0, t_0) = \varphi_{\Delta t}(x_0, t_0') =: \varphi_{\Delta t}(x_0).
\]

In other words, the state of the system only depends on the initial state and the time elapsed, but not on the initial time. In this case,

\[
\varphi_{\Delta t + \Delta t'}(x_0) = \varphi_{\Delta t'}(\varphi_{\Delta t}(x_0))
\]

and \( \varphi_0(x_0) = x_0 \), since the state should remain the same if it is given no time to change. This condition defines a *flow*.

**Definition 0.1** (Flow). A flow on a set \( X \) is a family of maps \( \varphi_t : X \to X \) such that \( \varphi_0 \) is the identity map and

\[
\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))
\]
for all $t, s \in \mathbb{R}$ and $x \in X$.

Suppose now that we have a measure space $(X, \mathcal{F}, \mu)$ (think of $\mathbb{R}^n$ with the standard volume measure) and that the flow is (measurable and) measure preserving. We may say in this case that the flow is conservative. To say that the flow is measure preserving means that for any (measurable) subset $A \subseteq X$ and $t \in \mathbb{R}$,

$$\mu(\varphi_t(A)) = \mu(A).$$

Note that this is equivalent to

$$\mathbb{1}_A \circ \varphi_t = \mathbb{1}_A.$$

(Check this!) Taking linear combinations (and limits), we obtain the equivalent characterization of measure invariance: for all (measurable, integrable) functions $f : X \to \mathbb{C}$,

$$\int_X f \circ \varphi_t \, d\mu = \int_X f \, d\mu.$$

The flow on $X$ then defines a family of operators on the Hilbert space $L^2(X, \mathcal{F}, \mu)$ given by composition of functions:

$$T_t f := f \circ \varphi_{-t}$$

and these operators are unitary:

$$\langle T_t f, T_t g \rangle = \int_X f \circ \varphi_{-t} g \circ \varphi_{-t} \, d\mu = \int_X (f g) \circ \varphi_t \, d\mu = \int_X f g \, d\mu = \langle f, g \rangle.$$

Note in addition that $T_{-t} = T_t^*$:

$$\langle T_{-t} f, g \rangle = \int_X f \circ \varphi_{-t} g \, d\mu = \int_X f g \circ \varphi_{-t} \, d\mu = \langle f, T_t g \rangle = \langle T_t^* f, g \rangle.$$

Since $T_t$ is unitary, we have a right to expect (with things properly stated, this would be Stone’s theorem) that there exists a self-adjoint operator $H$ on the Hilbert space such that

$$T_t = e^{-i t H}.$$  

We will refer to $H$ as the generator of the time evolution of the system or as the system Hamiltonian. Let us denote by $\mathcal{H}$ the Hilbert space $L^2(X, \mathcal{F}, \mu)$. If $X = \{1, \ldots, n\}$ is a finite set with $n$ elements, then $\mathcal{H}$ is simply $\mathbb{C}^n$ with the standard inner product and orthonormal basis $\{|1\rangle, \ldots, |n\rangle\}$. Given any $u \in \mathcal{H}$ we have that the vector $u_t := T_t u$ satisfies the differential equation

$$\frac{d u_t}{d t} = -i H u_t$$

since we can expect (at least in finite dimensions, but actually true fairly generally once the basic theory of self-adjoint operators in Hilbert spaces is properly spelled out) that

$$\frac{d}{d t} T_t u = \left(\frac{d}{d t} e^{-i t H}\right) u = -i H e^{-i t H} u = -i H T_t u.$$

The conclusion here is that, already classically, it is natural to think that the time evolution of a conservative deterministic system defines a unitary flow on a Hilbert space and that this evolution is generated by a self-adjoint operator. We take this as the starting point in the quantum setting.

We thus assume that the time evolution of quantum system with Hilbert space $\mathcal{H}$ is generated by a self-adjoint
operator \( H \), called the system’s Hamiltonian, and that \( H \) gives rise to a flow (or 1-parameter group) of unitary operators

\[ U_t = e^{-itH}. \]

If \( u \) is a unit vector in \( \mathcal{H} \) representing an initial state pure state, then \( u \) evolves in time according to

\[ u_t = U_t u \]

and \( u_t \) satisfies the Schrödinger equation

\[ \frac{du_t}{dt} = -iHu_t. \]

The state itself, \( \rho = |u\rangle \langle u| \), has a time evolution given by the family \( \rho_t \) which satisfies

\[ \frac{d\rho_t}{dt} = -i[H, \rho_t]. \]

We wish now to explore these ideas in the context of a single qubit system.

1. Let \( \mathcal{H} = \mathbb{C}^2 \). The self-adjoint matrices constitute a 4-dimensional space of \( 2 \times 2 \)-matrices spanned by the Pauli matrices

\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Thus any self-adjoint operator on \( \mathcal{H} \) can be written as

\[ H = a_0 \sigma_0 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3. \]

Let us write \( a \cdot \sigma := a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 \) so that \( H = a_0 I + a \cdot \sigma \). Suppose \( a \) is a unit vector in \( \mathbb{R}^3 \). Show that

\[ e^{-itH} = e^{-it a_0} [(\cos t) I - i(\sin t) a \cdot \sigma]. \]

(Exercise 4.18 in the notes.)

2. As an example, let \( H = \sigma_1 \). Find the unit vector \( u_t \) that solves the Schrödinger equation with the Hamiltonian \( H \) and \( u_0 = |0\rangle \).

3. Describe geometrically the states \( \rho_t = |u_t\rangle \langle u_t| \) obtained in the previous item as a subset of the Bloch sphere.

(You may use the result from Exercise 4.17 in the notes.)