Homework set 6 - due 12/01/22

Math 444

There are two parts to this assignment. In the first part we look at a class of problems in quantum probability theory that requires the use of symmetric and antisymmetric tensor products of Hilbert spaces. In physics, this is related to the subject of quantum statistics and the notions of fermions and bosons. For more on this topic (from a math perspective) see *An Introduction to Quantum Stochastic Calculus* by K.R. Parthasarathy (Springer, 1992, Chapter 2, Section 17) and *Indistinguishable Classical Particles* by Alexander Bach (Springer, 1997).

The second part has to do with the postulate about the dynamics of quantum states.

In the first part, most of the exercises, except number 6, already have answers. You should take those as a reading assignment.

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Quantum statistics. Consider a system consisting of n identical subsystems. We call these subsystems *particles*. To say that the particles are identical means that they agree in all their intrinsic (that is, state independent) properties. Effectively, for this assignment, this means that identical particles are associated to a same Hilbert space. If $\mathscr{H}$ denotes the Hilbert space of one subsystem, then the Hilbert space for the composite system is the n-fold tensor product

$$\mathscr{H}^{\otimes n} := \mathscr{H} \otimes \cdots \otimes \mathscr{H}.$$

Identical particles are said to be *indistinguishable* if they are in a state that is symmetric. To make sense of this statement, recall that S_n is the symmetric group in n symbols, i.e., the group of all permutations of the set $X = \{1, \dots, n\}$. Each permutation $\sigma \in S_n$ is a bijective function $\sigma : X \to X$. For each $\sigma \in S_n$, let $\Pi(\sigma)$ be the linear operator on $\mathscr{H}^{\otimes n}$ obtained by setting

$$\Pi(\sigma) u_1 \otimes \cdots \otimes u_n = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(n)}$$

on decomposable tensors, then extending $\Pi(\sigma)$ to all of $\mathscr{H}^{\otimes n}$ by linearity. Recall that Π is a group homomorphism from S_n to the group of unitary transformations on $\mathscr{H}^{\otimes n}$. This means that $\Pi(\sigma)$ is a unitary operator and $\Pi(\sigma\eta) = \Pi(\sigma)\Pi(\eta)$ for all $\sigma, \eta \in S_n$.

We can now state as a definition that the identical particles are indistinguishable if they are in a state (defined by the density operator) ρ which is symmetric in the sense that $\Pi(\sigma)\rho\Pi(\sigma^{-1}) = \rho$ for all $\sigma \in S_n$. For example, if ρ_0 is a state on $\mathscr{H}$ then $\rho_0^{\otimes n} := \rho_0 \otimes \cdots \otimes \rho_0$ is a symmetric state.

1. Let ρ be a symmetric state of a composite systems of n identical particles. Each particle's Hilbert space is $\mathscr{H}$. Let $P_1, \dots, P_n \in \mathscr{P}(\mathscr{H})$ be orthogonal projections on $\mathscr{H}$ representing events, and let the joint event be $P := P_1 \otimes \cdots \otimes P_n$. Define the permuted joint event $\Pi(\sigma)P\Pi(\sigma^{-1})$. (This is a composition of linear maps.)

   (a) Show that

   $$\Pi(\sigma)P\Pi(\sigma^{-1}) = P_{\sigma^{-1}(1)} \otimes \cdots \otimes P_{\sigma^{-1}(n)}.$$

(b) Show that the probability of any permuted event $\Pi(\sigma)P\Pi(\sigma^{-1})$ with respect to the symmetric state ρ is the same as the probability of P with respect to ρ. (Recall that the probability of the event P in the state ρ is $\mathrm{Tr}(\rho P)$.)

(c) Let P_s and P_a be the linear transformations on $\mathcal{H}^{\otimes n}$ defined by

$$P_s := \frac{1}{n!}\sum_{\sigma\in S_n}\Pi(\sigma), \quad P_a := \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\Pi(\sigma),$$

called the *symmetrization* and *antisymmetrization* operators. Show that $P_s, P_a \in \mathcal{P}(\mathcal{H}^{\otimes n})$. (Recall that the latter space consists of the orthogonal projections on the tensor product Hilbert space. Thus proving that $P \in \mathcal{P}(\mathcal{H}^{\otimes n})$ amounts to checking that $P^2 = P$ and that P is self-adjoint with respect to the inner product on $\mathcal{H}^{\otimes n}$.)

(d) Show that

$$\Pi(\sigma)P_s = P_s\Pi(\sigma) = P_s, \quad \Pi(\sigma)P_a = P_a\Pi(\sigma) = \mathrm{sgn}(\sigma)P_a$$

for all $\sigma \in S_n$.

(e) Let $\mathcal{H}_s^{\otimes n}$ be the range of P_s (i.e., the image of $\mathcal{H}^{\otimes n}$ under the transformation P_s). We call it the *symmetrized* tensor product. Similarly, define the *antisymmetrized* tensor product $\mathcal{H}_a^{\otimes n}$, the range of P_a. Show that the symmetric state ρ maps the symmetrized tensor product into itself and the antisymmetrized tensor product into itself.

(f) Let ρ be a symmetric state of n identical particles. Let $u \in \mathcal{H}^{\otimes n}$, $u \neq 0$, be an eigenvector of ρ corresponding to eigenvalue p. Show that the linear span of all $\Pi(\sigma)u$, $\sigma \in S_n$, is contained in the eigenspace of ρ for the eigenvalue p. (This amounts to checking that $\rho\Pi(\sigma)u = p\Pi(\sigma)u$ for all $\sigma \in S_n$.)

We say that the n particles are n identical *bosons* if their joint Hilbert space is $\mathcal{H}_s^{\otimes n}$ with a symmetric state. They are n identical *fermions* if their joint Hilbert space is $\mathcal{H}_a^{\otimes n}$ with a symmetric state.

Solution.

(a) Let us apply $\Pi(\sigma)P\Pi(\sigma^{-1})$ to a decomposable vector $u_1 \otimes \cdots \otimes u_n$:

$$\begin{aligned}\Pi(\sigma)P\Pi(\sigma^{-1})u_1 \otimes \cdots \otimes u_n &= \Pi(\sigma)Pu_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} \\ &= \Pi(\sigma)P_1u_{\sigma(1)} \otimes \cdots \otimes P_nu_{\sigma(n)} \\ &= P_{\sigma^{-1}(1)}u_1 \otimes \cdots \otimes P_{\sigma^{-1}(n)}u_n \\ &= P_{\sigma^{-1}(1)} \otimes \cdots \otimes P_{\sigma^{-1}(n)}u_1 \otimes \cdots \otimes u_n.\end{aligned}$$

As decomposable tensors of the form $u_1 \otimes \cdots \otimes u_n$ span $\mathcal{H}^{\otimes n}$, the claimed identity holds.

(b) If ρ is a symmetric state and P then, since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$,

$$\mathrm{Tr}(\rho\Pi(\sigma)P\Pi(\sigma^{-1})) = \mathrm{Tr}(\Pi(\sigma^{-1})\rho\Pi(\sigma)P) = \mathrm{Tr}(\rho P).$$

(c) I will show this for P_a. The same argument holds for P_s. First note that $\Pi(\sigma)$ satisfies

$$\Pi(\sigma)^* = \Pi(g^{-1}).$$

It suffices to verify this identity on separable tensors:

$$\begin{aligned}
\left\langle \Pi(\sigma)^* u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots v_n \right\rangle &= \langle u_1 \otimes \cdots \otimes u_n, \Pi(\sigma) v_1 \otimes \cdots v_n \rangle \\
&= \left\langle u_1 \otimes \cdots \otimes u_n, v_{\sigma^{-1}(1)} \otimes \cdots v_\sigma^{-1}(n) \right\rangle \\
&= \left\langle u_1, v_{\sigma^{-1}(1)} \right\rangle \cdots \left\langle u_n, v_{\sigma^{-1}(n)} \right\rangle \\
&= \left\langle u_{\sigma(1)}, v_1 \right\rangle \cdots \left\langle u_{\sigma(n)}, v_n \right\rangle \quad \text{(permuting the factors of this product according to } \sigma) \\
&= \left\langle u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}, v_1 \otimes \cdots \otimes v_n \right\rangle \\
&= \left\langle \Pi(\sigma^{-1}) u_1 \otimes \cdots \otimes u_1, v_1 \otimes \cdots \otimes v_n \right\rangle.
\end{aligned}$$

Therefore, since $\operatorname{sgn}(\sigma) = \operatorname{sign}(\sigma^{-1})$,

$$P_a^* = \frac{1}{n!} \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \Pi(\sigma) \right)^* = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \Pi(\sigma)^* = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \Pi(\sigma^{-1}) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1}) \Pi(\sigma^{-1}).$$

The group inverse $\sigma \mapsto \sigma^{-1}$ is a bijection. Therefore

$$\frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1}) \Pi(\sigma^{-1}) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \Pi(\sigma) = P_a.$$

This shows that $P_a^* = P_a$. Let us now check that $P_a^2 = P_a$.

$$\begin{aligned}
P_a P_a &= \left(\frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \Pi(\sigma) \right) \left(\frac{1}{n!} \sum_{\eta \in S_n} \operatorname{sgn}(\eta) \Pi(\eta) \right) \\
&= \frac{1}{n!} \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\eta \in S_n} \operatorname{sgn}(\sigma) \operatorname{sgn}(\eta) \Pi(\sigma) \Pi(\eta) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \left(\frac{1}{n!} \sum_{\eta \in S_n} \operatorname{sgn}(\sigma\eta) \Pi(\sigma\eta) \right) \quad \text{(sgn and } \Pi \text{ are homomorphisms)} \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} P_a \quad (\eta \mapsto \sigma\eta \text{ is a bijection}) \\
&= P_a \quad (S_n \text{ has order } n!)
\end{aligned}$$

(d) Let us check this for P_a. The same argument will apply to P_s. Since

$$\Pi(\sigma)\Pi(\eta) = \Pi(\sigma)\Pi(\eta), \quad \operatorname{sgn}(\sigma\eta) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\eta), \quad \operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1}),$$

then

$$\Pi(\sigma) P_a = \frac{1}{n!} \sum_{\eta \in S_n} \operatorname{sgn}(\eta) \Pi(\sigma\eta) = \frac{1}{n!} \sum_{\eta \in S_n} \operatorname{sgn}(\sigma^{-1}) \operatorname{sgn}(\sigma\eta) \Pi(\sigma\eta) = \operatorname{sgn}(\sigma) \frac{1}{n!} \sum_{\eta \in S_n} \operatorname{sgn}(\eta) \Pi(\eta) = \operatorname{sgn}(\sigma) P_a.$$

We have used that $\eta \mapsto \sigma\eta$ is a bijection. The other identities are shown in a similar way.

(e) It suffices to check that ρ sends elements of $\mathcal{H}_a^{\otimes n}$ of the form $P_a u$, $u \in \mathcal{H}^{\otimes n}$, to elements of the same form. But ρ commutes with each $\Pi(\sigma)$, therefore it commutes with P_a. Thus $\rho P_a u = P_a \rho u$. The same argument applies to P_s.

(f) This is immediate due to ρ commuting with $\Pi(\sigma)$.

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2. Let $\{e_1,\dots,e_N\}$ be an orthonormal basis for the N-dimensional Hilbert space $\mathcal{H}$. Let $r_1,\dots,r_N$ be nonnegative integers such that $r_1+\cdots+r_N=n$. It will be useful to employ the following notation:

$$e(r_1,\dots,r_N):=\underbrace{e_1\otimes\cdots\otimes e_1}_{r_1}\otimes\cdots\otimes\underbrace{e_i\otimes\cdots\otimes e_i}_{r_i}\otimes\cdots\otimes\underbrace{e_N\otimes\cdots\otimes e_N}_{r_N}.$$

Naturally, if $r_i=0$, the term e_i does not appear in the above product. The orthogonal projection to the one-dimensional subspace of $\mathcal{H}^{\otimes n}$ spanned by $e(r_1,\dots,r_N)$ will be denoted $|e(r_1,\dots,r_N)\rangle\langle e(r_1,\dots,r_N)|$.

(a) Show that

$$\left\{\left(\frac{n!}{r_1!\cdots r_N!}\right)^{1/2}P_s e(r_1,\dots,r_N): r_j\geq 0\ (j=1,\dots,N), r_1+\cdots+r_N=n\right\}$$

is an orthonormal basis for $\mathcal{H}_s^{\otimes n}$. Conclude that

$$\dim\mathcal{H}_s^{\otimes n}=\binom{N+n-1}{n}.$$

(b) Show that $\mathcal{H}_a^{\otimes n}=\{0\}$ if $n>N$. If $n\leq N$, show that

$$\left\{\sqrt{n!}P_a e(r_1,\dots,r_N): r_j\in\{0,1\}, r_1+\cdots+r_N=n\right\}$$

is an orthonormal basis for $\mathcal{H}_a^{\otimes n}$. Conclude that

$$\dim\mathcal{H}_a^{\otimes}=\binom{N}{n}.$$

Solution.

(a) Let us first check that these vectors are orthonormal. Let us define

$$a_{rs}:=\left\langle\left(\frac{n!}{r_1!\cdots r_N!}\right)^{1/2}P_s e(r_1,\dots,r_N),\left(\frac{n!}{s_1!\cdots s_N!}\right)^{1/2}P_s e(s_1,\dots,s_N)\right\rangle.$$

Then, since $P_s^*=P_s=P_s^2$,

$$\begin{aligned}
a_{rs}&=\frac{n!}{\sqrt{r_1!\cdots r_N!s_1!\cdots s_N!}}\langle e(r_1,\dots,r_N),P_s e(s_1,\dots,s_N)\rangle\\
&=\frac{1}{\sqrt{r_1!\cdots r_N!s_1!\cdots s_N!}}\sum_{\sigma\in S_n}\langle e(r_1,\dots,r_N),\Pi(\sigma)e(s_1,\dots,s_N)\rangle\\
&=\frac{1}{\sqrt{r_1!\cdots r_N!s_1!\cdots s_N!}}\sum_{\sigma\in S_n}\left\langle\underbrace{e_1\otimes\cdots\otimes e_1}_{r_1}\otimes\cdots\otimes\underbrace{e_N\otimes\cdots\otimes e_N}_{r_N},\Pi(\sigma)\underbrace{e_1\otimes\cdots\otimes e_1}_{s_1}\otimes\cdots\otimes\underbrace{e_N\otimes\cdots\otimes e_N}_{s_N}\right\rangle.
\end{aligned}$$

It is apparent from this last expression that it is nonzero only when $(r_1,\dots,r_N)=(s_1,\dots,s_N)$ and when σ permutes the factors in each block $e_i\otimes\cdots\otimes e_i$ among themselves. The number of permutations of this kind is $r_1!\cdots r_N!$ and each nonzero inner product in the above sum equals 1. Therefore $a_{r,s}=0$ if $r\neq s$ and $a_{r,r}=1$. Further note that the symmetrization of an arbitrary tensor of the form $e_{i_1}\otimes\cdots\otimes e_{i_n}$ equals

$P_s e(r_1,\dots,r_N)$ for some $(r_1,\dots,r_N)$ such that $r_1+\cdots+r_N=n$. Therefore the orthonormal family of tensors spans all of $\mathcal{H}_s^{\otimes n}$ and thus constitutes an orthonormal basis.

To show that the dimension is the given binomial expression, we need to count the ways we can separate the numbers $1,2,\dots,n$ into N bunches of consecutive numbers. We think of the bunches as obtained by placing $N-1$ walls between the numbers disposed in linear fashion. Thus let us imagine that there are a total of $n+N-1$ places disposed in a line and that each place is to be occupied by either a number from 1 to n (in this order) or by a wall. The number of bunches is then the number of ways of choosing $N-1$ places out of $n+N-1$. This is also equal to the number of ways of choosing n out of $N+n-1$, which is the given binomial expression.

(b) First note that $P_a e(r_1,\dots,r_N)=0$ if $r_i\geq 2$ for some i. In fact, let σ be an odd permutation that only permutes the factors in the block $e_i\otimes\cdots\otimes e_i$. Then

$$-P_a e(r_1,\dots,r_N)=\Pi(\sigma)P_a e(r_1,\dots,e_N)=P_a\Pi(\sigma)e(r_1,\dots,r_N)=P_a e(r_1,\dots,r_N).$$

This implies that $P_a e(r_1,\dots,r_N)=0$. Therefore the image of P_a is spanned by the vectors

$$\{P_a e(r_1,\dots,r_N),\ r_i\in\{0,1\},\ r_1+\cdots+r_N=n\}.$$

We can show that the vectors $\sqrt{n!}P_a e(r_1,\dots,r_N)$ are orthonormal by the same argument used in the previous item. Thus for the antisymmetrized tensor product Hilbert space to be non-zero we must have $N\geq n$. If we write these antisymmetrized vectors as $P_a e_{i_1}\otimes\cdots\otimes e_{i_n}$, $i_1<\cdots<i_n$, then the number of such vectors equals the number of ways of choosing n (ordered) indices among N numbers. But this is the given binomial coefficient.

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3. *Statistics of occupancy I: Maxwell-Boltzmann.* Let us first consider a classical probability problem. Suppose n identical balls are placed randomly into N urns. The urns are labeled $1,\dots,N$. The probability that any one of the balls will fall into urn j is assumed to be $p_j\geq 0$, so that $p_1+\cdots+p_N=1$. We wish to determine the probability of the event $E(r_1,\dots,r_N)$ of obtaining r_j balls in urn j for $j=1,\dots,N$. It is a basic fact from probability theory that the probability of this event is

$$\Pr(E(r_1,\dots,r_N))=\frac{n!}{r_1!\cdots r_N!}p_1^{r_1}\cdots p_N^{r_N}.$$

This is called the *multinomial distribution.* Let us derive this probability using the language of Hilbert spaces and linear operators.

The probability distribution for a single ball, written as a density operator, is

$$\rho_0=\sum_{j=1}^{N}p_j\left|j\right\rangle\left\langle j\right|.$$

Here, for a single ball, we employ the standard notation $|j\rangle=e_j$. Note that the probability of finding a given ball with density operator ρ_0 in urn s is the probability of the event $E_s:=|s\rangle\langle s|\in\mathcal{P}(\mathcal{H})$, which is

$$\Pr_{\rho_0}(E_s)=\operatorname{Tr}(\rho_0E_s)=\operatorname{Tr}\left(\sum_j p_j\left|j\right\rangle\left\langle j|s\right\rangle\left\langle s\right|\right)=\sum_j p_j|\langle j|s\rangle|^2=\sum_j p_j\delta_{js}=p_s.$$

We assume that the joint probability distribution for all the balls is given by $\rho = \rho_0^{\otimes n} = \rho_0 \otimes \cdots \otimes \rho_0$. To verify that the multinomial distribution indeed holds, we need to obtain the probability of the event $E(r_1, \dots, r_N)$, which may be written in Dirac notation as

$$E(r_1, \dots, r_N) = \sum_{|\{i: j_i = j\}| = r_j : j = 1, \dots, N} |j_1 \cdots j_n\rangle \langle j_1 \cdots j_n|,$$

where the summation is over all $(j_1, \dots, j_n) \in \{1, \dots, N\}^n$ such that each value $j \in \{1, \dots, N\}$ appears in $|j_1, \dots, j_n\rangle$ r_j times.

(a) Show that $E(r_1, \dots, r_N)$ is an orthogonal projection on $\mathcal{H}^{\otimes n}$ whose range has dimension $\frac{n!}{r_1! \cdots r_N!}$.

(b) Show that the probability of the event $E(r_1, \dots, r_N)$ in state $\rho_0^{\otimes n}$ is given by

$$\mathrm{Tr}\left(\rho_0^{\otimes n} E(r_1, \dots, r_N)\right) = \frac{n!}{r_1! \cdots r_N!} p_1^{r_1} \cdots p_N^{r_N}.$$

Solution.

(a) That $E(r_1, \dots, r_N)$ is an orthogonal projection is an immediate consequence of the fact that it is a sum of orthogonal projections that are mutually orthogonal. The dimension of the range is the dimension of the subspace of $\mathcal{H}^{\otimes n}$ spanned by the orthogonal unit vectors $|j_1 \cdots j_n\rangle$ such that each value $j \in \{1, \dots, N\}$ appears r_j times among the j_i. But the number of such vectors is the number of ways of distributing n (distinct) balls into N urns such that urn j has r_j balls. A standard counting method shows that this number is given by the multinomial coefficient

$$\binom{n}{r_1, \dots, r_N} = \frac{n!}{r_1! \cdots r_N!}.$$

(b) We have

$$\begin{aligned}
\mathrm{Tr}\left(\rho_0^{\otimes n} E(r_1, \dots, r_N)\right) &= \sum_{|\{i: j_i = j\}| = r_j : j = 1, \dots, N} \mathrm{Tr}\left(\rho_0^{\otimes n} |j_1 \cdots j_n\rangle \langle j_1 \cdots j_n|\right) \\
&= \sum_{|\{i: j_i = j\}| = r_j : j = 1, \dots, N} \langle j_1 \cdots j_n| \rho_0^{\otimes n} |j_1 \cdots j_n\rangle \\
&= \sum_{|\{i: j_i = j\}| = r_j : j = 1, \dots, N} p_{j_1} \cdots p_{j_n} \\
&= \frac{n!}{r_1! \cdots r_N!} p_1^{r_1} \cdots p_N^{r_N}.
\end{aligned}$$

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4. *Statistics of occupancy II: Bose-Einstein.* We suppose that the n balls are n identical bosons. The Hilbert space for the system is now $\mathcal{H}_s^{\otimes n}$. We take the probability distribution $\rho_0^{\otimes n}$ conditional on the event P_s. (Recall that P_s is the orthogonal projection to the symmetrized tensor product Hilbert space.) It makes sense to define the conditional probability distribution by the normalized density operator

$$\rho_s^{\otimes n} := \frac{\rho_0^{\otimes n}|_{\mathcal{H}_s^{\otimes n}}}{\mathrm{Tr}\left(\rho_0^{\otimes n} P_s\right)}.$$

(It should be clear that $\rho_0^{\otimes n}$ maps the symmetrized Hilbert space into itself.) Thus a system of n indistinguishable bosons may be defined by the quantum probability space

$$\left(\mathcal{H}_s^{\otimes n}, \mathcal{P}\left(\mathcal{H}_s^{\otimes n}\right), \rho_s^{\otimes n}\right).$$

We now wish to obtain the probability of the event that the n bosons are distributed over the N urns according to the occupation numbers $(r_1, \ldots, r_N)$. Imposing the symmetry on the event $E(r_1, \ldots, r_N)$, we have

$$E_s(r_1, \ldots, r_N) := P_s E(r_1, \ldots, r_N) P_s.$$

Note that this event corresponds to the orthogonal projection to the one-dimensional subspace of $\mathcal{H}_s^{\otimes n}$ spanned by the vector $P_s e(r_1, \ldots, r_N)$.

(a) Show that $E_s(r_1, \ldots, r_N)$ is the orthogonal projection to the one-dimensional subspace of $\mathcal{H}_s^{\otimes n}$ spanned by the tensor $\left(\frac{n!}{r_1! \cdots r_N!}\right)^{1/2} P_s |e(r_1, \ldots, r_N)\rangle$.

(b) Show that

$$c := \mathrm{Tr}\left(\rho_0^{\otimes n} P_s\right) = \sum_{s_1 + \cdots + s_N = n} p_1^{s_1} \cdots p_N^{s_N}.$$

(c) Show that the probability of the event $E_s(r_1, \ldots, r_N)$ in the bosonic state $\rho_s^{\otimes n} = c^{-1} \left.\rho_0^{\otimes n}\right|_{\mathcal{H}_s^{\otimes n}}$ is

$$\mathrm{Tr}\left(\rho_s^{\otimes n} E_s(r_1, \ldots, r_N)\right) = \frac{p_1^{r_1} \cdots p_N^{r_N}}{\sum_{s_1 + \cdots + s_N = n} p_1^{s_1} \cdots p_N^{s_N}}.$$

In this case, the indistinguishable particles are said to obey the *Bose-Einstein statistics.*

Solution.

(a) This follows from the observation that

$$\begin{aligned}
P_s E(r_1, \ldots, r_N) P_s &= \sum_{|\{i: j_i = j\}| = r_j : j = 1, \ldots, N} P_s |j_1 \cdots j_n\rangle \langle j_1 \cdots j_n| P_s \\
&= \sum_{|\{i: j_i = j\}| = r_j : j = 1, \ldots, N} P_s |e(r_1, \ldots, r_N)\rangle \langle e(r_1, \ldots, r_N)| P_s \\
&= \frac{n!}{r_1! \cdots r_N!} P_s |e(r_1, \ldots, r_N)\rangle \langle e(r_1, \ldots, r_N)| P_s,
\end{aligned}$$

which is the orthogonal projection to the one-dimensional subspace generated by the unit vector

$$\sqrt{\frac{n!}{r_1! \cdots r_N!}} P_s |e(r_1, \ldots, r_N)\rangle.$$

(b) We have, using the result of the first item:

$$\begin{aligned}
\mathrm{Tr}\left(\rho_0^{\otimes n} P_s\right) &= \sum_{s_1+\cdots+s_N=n} \mathrm{Tr}\left(\rho_0^{\otimes n} P_s E(s_1,\ldots,s_N) P_s\right) \\
&= \sum_{s_1+\cdots+s_N=n} \frac{n!}{s_1!\cdots s_N!} \mathrm{Tr}\left(\rho_0^{\otimes n} P_s \left|e(s_1,\ldots,s_N)\right\rangle \left\langle e(s_1,\ldots,s_N)\right| P_s\right) \\
&= \sum_{s_1+\cdots+s_N=n} \frac{n!}{s_1!\cdots s_N!} \left\langle e(s_1,\ldots,s_N)\right| \rho_0^{\otimes n} P_s \left|e(s_1,\ldots,s_N)\right\rangle \\
&= \sum_{s_1+\cdots+s_N=n} \frac{n!}{s_1!\cdots s_N!} p_1^{s_1}\cdots p_N^{s_N} \left\langle e(s_1,\ldots,s_N)\right| P_s \left|e(s_1,\ldots,s_N)\right\rangle \\
&= \sum_{s_1+\cdots+s_N=n} \frac{1}{s_1!\cdots s_N!} p_1^{s_1}\cdots p_N^{s_N} \sum_{\sigma\in S_n} \left\langle e(s_1,\ldots,s_N)\right| \Pi(\sigma) \left|e(s_1,\ldots,s_N)\right\rangle \\
&= \sum_{s_1+\cdots+s_N=n} \frac{1}{s_1!\cdots s_N!} p_1^{s_1}\cdots p_N^{s_N} s_1!\cdots s_N! \\
&= \sum_{s_1+\cdots+s_N=n} p_1^{s_1}\cdots p_N^{s_N}.
\end{aligned}$$

(c) The calculation of the previous item already shows that

$$\mathrm{Tr}\left(\rho_0^{\otimes n} E_s(r_1,\ldots,r_N)\right) = p_1^{r_1}\cdots p_N^{r_N}.$$

Therefore,

$$\mathrm{Tr}\left(\rho_s^{\otimes n} E_s(r_1,\ldots,r_N)\right) = \frac{p_1^{r_1}\cdots p_N^{r_N}}{\sum_{s_1+\cdots+s_N=n} p_1^{s_1}\cdots p_N^{s_N}}$$

as claimed.

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5. *Statistics of occupancy III: Fermi-Dirac.* Let the Hilbert space now be the antisymmetrized tensor product $\mathcal{H}_a^{\otimes n}$. We define the state $\rho_a^{\otimes n}$ as

$$\rho_a^{\otimes n} := \frac{\rho_0^{\otimes n}\big|_{\mathcal{H}_a^{\otimes n}}}{\mathrm{Tr}\left(\rho_0^{\otimes n} E_a\right)}.$$

A system of n indistinguishable fermions may be defined by the quantum probability space

$$\left(\mathcal{H}_a^{\otimes n}, \mathcal{P}\left(\mathcal{H}_a^{\otimes n}\right), \rho_a^{\otimes n}\right).$$

We now wish to obtain the probability of the event that the n fermions are distributed over the N urns according to the occupation numbers $(r_1,\ldots,r_N)$. Imposing the symmetry on the event $E(r_1,\ldots,r_N)$, we have

$$E_a(r_1,\ldots,r_N) := P_a E(r_1,\ldots,r_N) P_a.$$

(a) Show that $E_a(r_1,\ldots,r_N)$ is the orthogonal projection to the one-dimensional subspace of $\mathcal{H}_a^{\otimes n}$ spanned by the vector $\sqrt{n!}P_a\left|e(r_1,\ldots,r_N)\right\rangle$ where $r_j \in \{0,1\}, r_1+\cdots+r_N = n$. Alternatively, we may write this vector as

$$\sqrt{n!}P_a\left|j_1\ldots j_n\right\rangle, \quad j_1 < j_2 < \cdots < j_n.$$

(b) Show that

$$c := \mathrm{Tr}\left(\rho_0^{\otimes n} P_a\right) = \sum_{1\le j_1<\cdots<j_n\le N} p_{j_1}\cdots p_{j_n}.$$

(c) Show that the probability of the event $E_a(r_1,\dots,r_N)$ in the fermionic state $\rho_a^{\otimes n} = c^{-1}\ \rho_0^{\otimes n}\big|_{\mathcal{H}_a^{\otimes n}}$ is

$$\mathrm{Tr}\left(\rho_0^{\otimes n} E_a(r_1,\dots,r_N)\right) = \frac{p_{j_1}\cdots p_{j_n}}{\sum_{1\le i_1<\cdots<i_n\le N} p_{i_1}\cdots p_{i_n}}.$$

In this case, the indistinguishable particles are said to obey the *Fermi-Dirac statistics.*

Solution.

(a) We have

$$\begin{aligned}
E_a(r_1,\dots,r_N) &= P_a E(r_1,\dots,r_N) P_a \\
&= \sum_{|\{i:j_i=j\}|=r_j:j=1,\dots,N} P_a |j_1\cdots j_n\rangle\langle j_1\cdots j_n| P_a \\
&= \sum_{|\{i:j_i=j\}|=r_j:j=1,\dots,N} P_a |e(r_1,\dots,r_N)\rangle\langle e(r_1,\dots,e_N)| P_a.
\end{aligned}$$

Note that $P_a e(r_1,\dots,r_N) = 0$ if $r_j \ge 2$ for some j. In fact, if $e_i\otimes\cdots\otimes e_i$ is a block in $e(r_1,\dots,r_N)$ with at least 2 factors and $\sigma\in S_n$ is a transposition which exchanges two equal factors in this tensor product, then $\Pi(\sigma)e(r_1,\dots,r_N) = e(r_1,\dots,r_N)$. Since also $\Pi(\sigma)P_a = \mathrm{sgn}(\sigma)P_a$, we have

$$P_a e(r_1,\dots,r_N) = P_a\Pi(\sigma)e(r_1,\dots,r_N) = \Pi(\sigma)P_a e(r_1,\dots,r_N) = -P_a e(r_1,\dots,r_N).$$

Therefore

$$E_a(r_1,\dots,r_N) = \sum_{|\{i:j_i=j\}|=r_j\in\{0,1\}:j=1,\dots,N} P_a |e(r_1,\dots,r_N)\rangle\langle e(r_1,\dots,e_N)| P_a.$$

The number of terms in the above summation is the number of n-tuples $(j_1,\dots,j_n)$ consisting of distinct j_i, one for each urn that are occupied (corresponding to a $r_j = 1$). This is the number of all permutations of the entries of $(j_1,\dots,j_n)$, which is $n!$. Therefore,

$$E_a(r_1,\dots,r_N) = n! P_a |e(r_1,\dots,r_N)\rangle\langle e(r_1,\dots,e_N)| P_a = \left(\sqrt{n!}P_a|e(r_1,\dots,r_N)\rangle\right)\left(\sqrt{n!}\langle e(r_1,\dots,r_N)|\right)P_a.$$

(b) We have

$$\begin{aligned}
\mathrm{Tr}\left(\rho_0^{\otimes n}E_a(r_1,\dots,r_N)\right) &= n!\,\mathrm{Tr}\left(\rho_0^{\otimes n}P_a|j_1\cdots j_n\rangle\langle j_1\cdots j_n|P_a\right) \\
&= n!\langle j_1\cdots j_n|P_a\rho_0^{n\otimes}|j_1\cdots j_n\rangle \\
&= p_{j_1}\cdots p_{j_n} n!\langle j_1\cdots j_n|P_a|j_1\cdots j_n\rangle \\
&= p_{j_1}\cdots p_{j_n}.
\end{aligned}$$

The probability of P_a in state $\rho_0^{\otimes n}$ is the sum of the above probabilities over all the $i_1<\cdots<i_n$:

$$\mathrm{Tr}\left(\rho_0^{\otimes n}P_a\right) = \sum_{1\le i_1<\cdots<i_n\le N} p_{i_1}\cdots p_{i_n}.$$

Thus we conclude that

$$\mathrm{Tr}\left(\rho_0^{\otimes n} E_a(r_1,\dots,r_N)\right) = \frac{p_{j_1}\cdots p_{j_n}}{\sum_{1\le i_1<\cdots<i_n\le N} p_{i_1}\cdots p_{i_n}}.$$

◇

6. In order to compare the above three distributions, consider the case of n particles and 2 urns. Let the single particle state be

$$\rho_0 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|.$$

Show the following:

   (a) According to the Maxwell-Boltzmann statistics the number of particles in urn 1 has a binomial distribution given by

$$\Pr\big(\text{urn 1 has } k \text{ particles}\big) = \binom{n}{k} 2^{-n}, \quad 0 \le k \le n.$$

   In particular, the probability that all the balls occupy the first urn is 2^{-n}.

   (b) According to the Bose-Einstein statistics

$$\Pr\big(\text{urn 1 has } k \text{ particles}\big) = \frac{1}{n+1}, \quad 0 \le k \le n.$$

   In particular, the probability that all the balls occupy the first urn is $1/(n+1)$.

   (c) According to the Fermi-Dirac statistics it is not possible to have more than two particles when there are only two urns. Two particles cannot occupy the same urn. This is the so-called *Pauli exclusion principle*.

◇

Time evolution of quantum states and the Schrödinger equation. Let us begin with a few remarks about the dynamics of states in the classical setting. Let X be a set whose elements represent the classical (pure) states of the system. Here, we mean by a *state* something more than we meant previously. We suppose that the system can evolve in time in a deterministic way and that the state contains all the information we need in principle to uniquely determine the system's future (and past) at any given time. Thus there exists a map φ such that, if $x_0 \in X$ is the state of the system at time t_0, then at the time $t_0 + \Delta t$ the state will be $\varphi_{\Delta t}(x_0, t_0)$. At the time $t_0 + \Delta t + \Delta t'$, the system will evolve from its state at time $t_0 + \Delta t$ to the new state $\varphi_{\Delta t'}(\varphi_{\Delta t}(x_0, t_0), t_0 + \Delta t)$. By uniqueness of the time evolution (determinism), this is the same state we obtain by starting from x_0 at t_0 and letting the system evolve over the time interval $\Delta t + \Delta t'$. Thus

$$\varphi_{\Delta t + \Delta t'}(x_0, t_0) = \varphi_{\Delta t'}(\varphi_{\Delta t}(x_0, t_0), t_0 + \Delta t).$$

For simplicity, we will only consider the case in which the law governing the evolution of the system does not change in time (time homogeneity). This means that

$$\varphi_{\Delta t}(x_0, t_0) = \varphi_{\Delta t}(x_0, t_0') =: \varphi_{\Delta t}(x_0).$$

In other words, the state of the system only depends on the initial state and the time elapsed, but not on the initial time. In this case,

$$\varphi_{\Delta t + \Delta t'}(x_0) = \varphi_{\Delta t'}(\varphi_{\Delta t}(x_0))$$

and $\varphi_0(x_0) = x_0$, since the state should remain the same if it is given no time to change. This condition defines a *flow*.

Definition 0.1 (Flow)**.** *A flow on a set X is a family of maps $\varphi_t : X \to X$ such that φ_0 is the identity map and*

$$\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$$

for all $t, s \in \mathbb{R}$ *and* $x \in X$.

Suppose now that we have a measure space $(X, \mathcal{F}, \mu)$ (think of $\mathbb{R}^n$ with the standard volume measure) and that the flow is (measurable and) measure preserving. We may say in this case that the flow is *conservative*. To say that the flow is measure preserving means that for any (measurable) subset $A \subseteq X$ and $t \in \mathbb{R}$,

$$\mu(\varphi_t(A)) = \mu(A).$$

Note that this is equivalent to

$$\mathbb{1}_A \circ \varphi_t = \mathbb{1}_A.$$

(Check this!) Taking linear combinations (and limits), we obtain the equivalent characterization of measure invariance: for all (measurable, integrable) functions $f : X \to \mathbb{C}$,

$$\int_X f \circ \varphi_t \, d\mu = \int_X f \, d\mu.$$

The flow on X then defines a family of operators on the Hilbert space $L^2(X, \mathcal{F}, \mu)$ given by composition of functions:

$$T_t f := f \circ \varphi_{-t}$$

and these operators are unitary:

$$\langle T_t f, T_t g\rangle = \int_X \overline{f \circ \varphi_{-t}} g \circ \varphi_{-t} \, d\mu = \int_X \left(\overline{f} g\right) \circ \varphi_t \, d\mu = \int_X \overline{f} g \, d\mu = \langle f, g\rangle.$$

Note in addition that $T_{-t} = T_t^*$:

$$\langle T_{-t} f, g\rangle = \int_X \overline{f \circ \varphi_t} g \, d\mu = \int_X \overline{f} g \circ \varphi_{-t} \, d\mu = \langle f, T_t g\rangle = \langle T_t^* f, g\rangle.$$

Since T_t is unitary, we have a right to expect (with things properly stated, this would be Stone's theorem) that there exists a self-adjoint operator H on the Hilbert space such that

$$T_t = e^{-itH}.$$

We will refer to H as the *generator* of the time evolution of the system or as the system *Hamiltonian*. Let us denote by $\mathcal{H}$ the Hilbert space $L^2(X, \mathcal{H}, \mu)$. If $X = \{1, \ldots, n\}$ is a finite set with n elements, then $\mathcal{H}$ is simply $\mathbb{C}^n$ with the standard inner product and orthonormal basis $\{|1\rangle, \ldots, |n\rangle\}$. Given any $u \in \mathcal{H}$ we have that the vector $u_t := T_t u$ satisfies the differential equation

$$\frac{du_t}{dt} = -iHu_t$$

since we can expect (at least in finite dimensions, but actually true fairly generally once the basic theory of self-adjoint operators in Hilbert spaces is properly spelled out) that

$$\frac{d}{dt} T_t u = \frac{d}{dt} e^{-itH} u = -iHe^{-itH} u = -iHT_t u.$$

The conclusion here is that, already classically, it is natural to think that the time evolution of a conservative deterministic system defines a unitary flow on a Hilbert space and that this evolution is generated by a self-adjoint operator. We take this as the starting point in the quantum setting.

We thus assume that the time evolution of quantum system with Hilbert space $\mathcal{H}$ is generated by a self-adjoint operator H, called the system's *Hamiltonian*, and that H gives rise to a flow (or 1-parameter group) of unitary operators

$$U_t = e^{-itH}.$$

If u is a unit vector in $\mathscr{H}$ representing an initial state pure state, then u evolves in time according to

$$u_t = U_t u$$

and u_t satisfies the *Schr"odinger equation*

$$\frac{du_t}{dt} = -iHu_t.$$

The state itself, $\rho = |u\rangle\langle u|$, has a time evolution given by the family ρ_t which satisfies

$$\frac{d\rho_t}{dt} = \frac{d}{dt}|u_t\rangle\langle u_t| = \frac{d}{dt}e^{-itH}|u\rangle\langle u|e^{itH} = -iH|u_t\rangle\langle u_t| + |u_t\rangle\langle u_t|iH = -i[H,\rho_t].$$

Thus

$$\frac{d\rho_t}{dt} = -i[H,\rho_t].$$

We wish now to explore these ideas in the context of a single qubit system.

1. Let $\mathscr{H} = \mathbb{C}^2$. The self-adjoint matrices constitute a 4-dimensional space of 2×2-matrices spanned by the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Thus any self-adjoint operator on $\mathscr{H}$ can be written as

$$H = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3.$$

   Let us write $\mathbf{a}\cdot\sigma := a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$ so that $H = a_0 I + \mathbf{a}\cdot\sigma$. Suppose $\mathbf{a}$ is a unit vector in $\mathbb{R}^3$. Show that

$$e^{-itH} = e^{-ita_0}\left[(\cos t)I - i(\sin t)\mathbf{a}\cdot\sigma\right].$$

   (Exercise 4.18 in the notes.)

2. As an example, let $H = \sigma_1$. Find the unit vector u_t that solves the Schrödinger equation with the Hamiltonian H and $u_0 = |0\rangle$.

3. Describe geometrically the states $\rho_t = |u_t\rangle\langle u_t|$ obtained in the previous item as a subset of the Block sphere. (You may use the result from Exercise 4.17 in the notes.)