Homework set 3 - due 02/11/2022

Math 495 – Renato Feres

Problems

Work on all the exercises in this homework set. You will be asked to turn in for grading Exercises 1, 2, 4.

1. (Textbook, Exercise 3.2.) A stochastic matrix is called *doubly stochastic* if its rows and columns sum to 1. Show that a Markov chain whose transition matrix is doubly stochastic has a stationary distribution, which is uniform on the state space. (Assume the state space is a finite set.)

2. (Textbook, Exercise 3.4. Slightly modified.) Consider a Markov chain with transition matrix

$$P = \begin{pmatrix} 1-a & a & 0 \\ 0 & 1-b & b \\ c & 0 & 1-c \end{pmatrix},$$

   where $0 < a, b, c < 1$.

   (a) Is P a regular matrix? If so, prove it.

   (b) Find a stationary distribution.

   (c) Is the stationary distribution unique? If so, explain why (by referring to some theorem) without doing any calculation.

3. (Textbook, Exercise 3.7.) A Markov chain has n states. If the chain is at state k, a coin is flipped, whose heads probability is p. We suppose that $0 < p < 1$. If the coin lands heads, the chain stays at k. If the coin lands tails, the chain moves to a different state uniformly at random.

   (a) Exhibit the transition matrix.

   (b) Find a stationary distribution.

   (c) Is the stationary distribution you found unique?

4. (Textbook, Exercise 3.8.) Let

$$P_1 = \begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1/5 & 4/5 \\ 4/5 & 1/5 \end{pmatrix}.$$

   Consider a Markov chain on four states whose transition matrix is given by the block matrix

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

   (a) Does the Markov chain have a unique stationary distribution? If so, find it.

(b) Does $\lim_{n\to\infty} P^n$ exist? If so, find it.

(c) Does the Markov chain have a limiting distribution? If so, find it.

5. Consider the Markov chain with state space $S = \{0, \ldots, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0.1 & 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.25 & 0.25 & 0 & 0.25 & 0.25 \\ 0.7 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{pmatrix}.$$

(a) Draw a states and transitions graph.

(b) What are the communication classes?

(c) Which communication classes are recurrent and which are transient?

(d) Suppose the system starts in state 2. What is the probability that it will be in state 2 at some large time?

(e) Suppose the system starts in state 5. What is the probability that it will be in state 5 at some large time?

(f) Numerically obtain P^{500}. Are your answers to d and e supported by the values of the entries of this matrix?