Homework set 4 - due 02/18/22

Math 495 – Renato Feres

Solve all the 5 problems. You will be asked to turn in numbers 3, 4, and 5.

Problems

1. (Text, Exercise 3.19, page 148.) Consider random walk on the graph in Figure 1. Use *first-step analysis* (see page 105 of the textbook) to find the expected time to hit d for the walk started in a. (*Hint:* By exploiting symmetries in the graph, the solution can be found by solving a 3×3 linear system.)

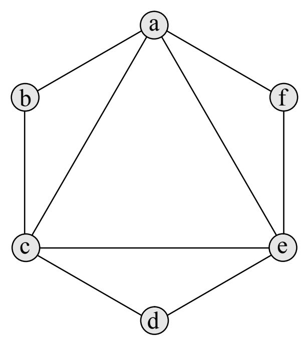

Figure 1: Diagram for Exercise 1.

If we write $F(x)$ for the expected time to hit d starting the random walk at x, in addition to $F(a)$, what are $F(b)$ and $F(c)$?

2. (Text, Exercise 3.23, page 149.) Consider a k-state Markov chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ k-2 \\ k-1 \\ k \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & k-2 & k-1 & k \end{array} \\ \begin{pmatrix} 1/k & 1/k & 1/k & \cdots & 1/k & 1/k & 1/k \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

   (a) Draw the transition diagram.

   (b) Show that the chain is ergodic. (See Section 3.6, beginning on page 109 of the textbook.)

(c) Find the limiting distribution.

3. (Text, Exercise 3.28, page 150.) Consider a Markov chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 1/2 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4 \end{pmatrix} \end{array}.$$

(a) Identify the communication classes. (Drawing a nice transition diagram helps.)

(b) Classify the states as recurrent or transient.

(c) Rewrite the transition matrix P in block form adapted to the canonical decomposition of states as described at the bottom of page 101 in the textbook.

(d) For all i and j, determine $\lim_{n\to\infty} P^n_{ij}$ without using technology. (See page 98 of the textbook.)

4. Suppose we flip a fair coin repeatedly until we have flipped four consecutive heads, then stop.

(a) Give a Markov chain (by drawing its transition diagram with the transition probabilities indicated next to the arrows) that represents this random experiment. (Hint: consider a Markov chain with state space $S = \{0,1,2,3,4\}$. The state at any given time is the number of consecutive heads since the last time tail came up. The state is reset to 0 at each occurrence of tail and 4 is an absorbing state.)

(b) Write down the transitions matrix P.

(c) Describe the communication classes and indicate which are transient and which are recurrent.

(d) What is the expected number of flips that are needed? (This can be done using the one-step analysis as in Exercise 1 or by applying the result at the bottom of page 126 of the textbook.)

(e) Do a computer simulation to verify that your answer is reasonable. I suggest the following: modify the Markov chain program given in the preliminaries section of this assignment so that it gives the number of steps till a given state is reached (in this case, the state 4) for each run of the chain. Then run the chain, starting at 0, a large number of times so as to get a large number of sample values of the random variable T_4 (the number of steps to reach state 4). Then compute the sample mean of T_4. This mean should, by the law of large numbers, approximate the expected value we want.

5. Consider simple random walk on the graph below. (Recall that a simple random walk on a graph is the Markov chain which at each time moves to an adjacent vertex, each adjacent vertex having the same probability.)

(a) Draw the transitions diagram with the values of the transition probabilities next to each arrow.

(b) Write the transitions matrix P. (States A, B, etc. correspond to row and column indices $1, 2$, etc.)

(c) In the long run, about what fraction of time is spent in vertex A?

(d) Suppose a walker starts in vertex A. What is the expected number of steps until the walker returns to A?

(e) Suppose a walker starts in vertex C. What is the expected number of visits to B before reaching A?

(f) Suppose a walker starts in vertex D. What is the probability that the walker reaches A before reaching B?

(g) Again assume the walker starts in C. What is the expected number of steps until the walker reaches A?

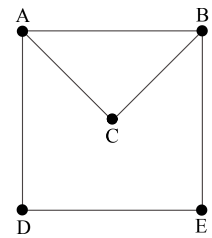

Figure 2: Graph for Exercise 5.

Topics involved in this exercise: (1) Proportion of time in a given recurrent state (page 78); (2) Expected return time to a recurrent state (Theorem 3.6, page 103); (3) Expected number of visits to a state before reaching another (by forcing the latter state to be absorbing, we can apply Theorem 3.11 on page 125); (4) Probability to be absorbed by a given state (Absorbing Markov chains, page 126, highlighted box).