# Homework set 5 - due 02/25/2022

Math 495 – Renato Feres

**Problems**

As always, work on all the exercises. You'll be asked to submit for grading numbers 1, 3, 4 and 5.

1. (Text, Exercise 4.2, page 175.) A random variable $X$ is said to be *Poisson* with rate parameter $\lambda > 0$ (denoted $X \sim \text{Poisson}(\lambda)$) if it takes values in the set $\{0, 1, 2, \dots\}$ and

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

   For this problem, read section 4.3 (Probability Generating Functions) of the textbook.

   (a) Find the probability generating function (pgf) of a Poisson random variable with parameter $\lambda$.

   (b) Use the pgf to find the mean of the Poisson distribution.

   (c) Use the pgf to find the variance of the Poisson distribution.

2. (Text, Exercise 4.3, page 175.) Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Assume that $X$ and $Y$ are independent. Use probability generating functions to find the distribution of $X + Y$.

3. (Text, Exercise 4.6, page 175.) A random variable $X$ is said to be *Bernoulli* with *success probability* $p$ if it takes values in the set $\{0, 1\}$ and $P(X = 1) = p$, $P(X = 0) = 1 - p$. Let $X_1, X_2, \dots$ be a sequence of i.i.d. (independent and identically distributed) Bernoulli random variables with parameter $p$. Let $N$ be a Poisson random variable with parameter $\lambda$, which is independent of the $X_i$

   (a) Find the probability generating function of $X_i$.

   (b) Find the probability generating function of $Z = \sum_{i=1}^{N} X_i$.

   (c) Use the first part to identify the probability distribution of $Z$.

4. (Text, Exercise 4.12, page 176.) A branching process has offspring distribution $\mathbf{a} = (1/4, 1/4, 1/2)$. Find the following:

   (a) $\mu$, the mean value of the offspring distribution;

   (b) $G(s)$, the moment generating function of the offspring distribution;

   (c) The extinction probability;

   (d) $P(Z_2 = 0)$, the probability of extinction by the second generation. (See the proof of Theorem 4.2, page 173.)

5. (Text, Exercise 4.29, page 179.) Simulate the branching process of the previous exercise. Use your simulations to estimate the extinction probability $e$. (Note: The text asks you to use a certain `branching.R` script,

but I don't know where to find it. I recommend that you write your own program, which is a more illuminating and rewarding approach.) Here are some suggestions: The number of offspring random variable $X$ can be simulated by using the `sample` function in R. Set the possible number of offspring of one individual as an array `Noffs=c(0,1,2)` and the probability distribution as another array `Poffs=c(1/4,1/4,1/2)`. Then the random number of offspring of one individual is given by `sample(Noffs,1,replace=TRUE,Poffs)`. An array of random number of offspring of 10 individuals reproducing independently can be obtained with the command `sample(Noffs,10,replace=TRUE,Poffs)` and `sum(sample(Noffs,10,replace=TRUE,Poffs))` gives the total number of offspring of those 10 individuals. If we want to follow the changing population number over, say 15 generations, we could do as follows:

```
M    = 15
Z    = 0*c(1:M)
Z[1] = 1
for (j in 2:M) {
  Z[j] = sum(sample(N_offs,Z[j-1],replace=TRUE,P_offs))
}
```

Here are typical sequences you'd get by running this small script a few times:

```
[1] 1  2  2  2  3  6  5  8 12 15 24 24 34 38 52
[1] 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0
[1] 1   2   4   5   8  15  18  24  26  30  40  48  65  83 103
[1] 1 2 2 0 0 0 0 0 0 0 0 0 0 0 0
```

A little experimentation seems to indicate to me that by the 15th generation the population size is either fairly large or extinct. Thus one way of finding the extinction probability is to repeat this experiment many times (say, by wrapping the above script in a `for` loop; I used 500000 trials of this experiment, which took about half a minute to compute) and then counting the proportion of times `Z[M]==0`. e $e = 0.5$.