Homework set 6 - due 03/04/22

Math 495 – Renato Feres

Problems

Do all the exercises. You will be asked to submit for grading numbers 1, 2 and 3.

1. (Text, Exercise 5.1, page 219.) Four out of every five trucks on the highway are followed by a car, while only one out of every four cars is followed by a truck. At a toll booth, cars pay \$1.50 and trucks pay \$ 5.00.
   (a) Give the transition matrix of a Markov chain with two states $\{\text{Car}, \text{Truck}\}$ that describes the sequence of types of vehicles passing through the tollbooth.
   (b) Find the (unique) stationary distribution.
   (c) If 1,000 vehicles pass through the tollbooth in one day, how much toll is collected?
   (d) Explain how the Law of Large Numbers for Markov chains is used in this problem.

2. (Text, Exercise 5.2, page 219.) Consider simple random walk on $\{0, 1, \cdots, k\}$ with reflecting boundaries at 0 and k, that is, random walk on the path from 0 to k. A random walker earns \$$k$ every time the walk reaches 0 or k, but loses \$1 at each internal vertex (from 1 to $k-1$.)
   (a) Write down the transition matrix for this random walk.
   (b) Is this Markov chain ergodic?
   (c) Find the stationary distribution. Is it unique?
   (d) In $10,000$ steps of the walk, how much, on average, will be gained?
   (e) Explain how the Law of Large Numbers for Markov chains was used in your calculation.
   (f) Do a Markov chain simulation of this experiment to provide numerical evidence for the result obtained in the previous item.

3. (Text, Exercise 5.6, page 220.) Recall a random variable X is Poisson with parameter λ if X has values in $\{0, 1, 2, \dots\}$ and
$$P(X = n) = \pi_n = \frac{\lambda^n}{n!} e^{-\lambda}.$$
   Show how to generate a Poisson random variable with parameter λ using the Metropolis-Hastings algorithm. Use simple symmetric random walk (reflecting at 0) as the proposal distribution. (This is the random walk on $\{0, 1, 2, \dots\}$ with transition probability $1/2$ for the transition from i to $i+1$ and from $i+1$ to i for $i \geq 1$, and probability 1 for the transition from 0 to 1.)
   (a) Write down the acceptance function for the Metropolis-Hastings algorithm. (See Section 5.2, page 187.)
   (b) Write down the transition probabilities for the Metropolis-Hastings Markov chain.

(c) Run the Markov chain for a large number of steps and produce a histogram of the result. Compare it with the Poisson distribution.

A few suggestions for the last part: assume $\lambda = 5$, make your sample (length of the Markov chain) large (say $N = 100000$; it shouldn't take too long to compute) and set your histogram break points appropriately so that bin midpoints are integers (something like `hist(X,breaks=seq(-0.5,20+0.5,by=1),freq=FALSE)`). In order to compare with the actual Poisson distribution, you can do the following after plotting the histogram:

```
m = dpois(c(0:20),lambda)
points(c(0:20),m)
```

This creates a vector `m` with the Poisson distribution values (on the set of integers from 0 to 20 with the chosen parameter λ) then plots these values on top of the histogram.

When writing the transition matrix P, it is more convenient to express it not as a table (the actual matrix) but as a set of cases. For example, the transition matrix for the symmetric random walk on $\{0,1,2,\dots\}$ can be written as

$$T_{ij} = \begin{cases} \frac{1}{2} & \text{if } |i-j| = 1 \text{ and } i \neq 0 \\ 1 & \text{if } i = 0 \text{ and } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

4. (Text, Exercise 5.17, page 222.) Random variables X and N have joint distribution, defined up to a constant of proportionality,

$$f(x,n) \propto \frac{e^{-3x}x^n}{n!}, \quad \text{for } n = 0,1,2,\dots \text{ and } x > 0.$$

Note that X is continuous and N is discrete.

(a) The conditional distributions of X given $N = n$, and of N given $X = x$ are standard. What are they? (See the textbook, Appendix C.)

(b) Write down in a concise and clear way the Gibbs sampler algorithm from this distribution.

(c) Implement the Gibbs sampler in R. (Section 6 in Appendix A is useful here.)

(d) Use your simulation to estimate (i) $P(X^2 < N)$ and (ii) $E(XN)$.

(e) Obtain analytically the exact value of $E(XN)$. First obtain the normalization constant C that makes

$$f(x,n) = C\frac{e^{-3x}x^n}{n!}$$

a probability distribution. Then calculate

$$E(XN) = \int_0^\infty \sum_{n=0}^\infty xnf(x,n)\,dx.$$