Homework set 2 – Due 09/14/18

Math 497 – Renato Feres

1. **The dual space.** Let $V$ be a real or complex finite dimensional vector space, not necessarily equipped with an inner product. I will use $F$ for either $\mathbb{R}$ or $\mathbb{C}$. Let $V^*$ denote the set of all the linear maps $\alpha : V \to F$. Convince yourself (it is not necessary to prove it here) that $V^*$ is also an $F$-vector space under the operations of sum and multiplication by scalars of linear maps. If $B = \{u_1, \ldots, u_n\}$ is any basis of $V$, we define $B^* = \{\mu_1, \ldots, \mu_n\} \subset V^*$ so that for each $i$

$$
\mu_i(u_j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j.
\end{cases}
$$

for all $j$.

(a) Show that $B^*$ is a basis of $V^*$. It is called the dual basis. (Suggestion: to show that $B^*$ linearly spans $V$, check that $f = \sum_i f(u_i) \mu_i$ for all $f \in V^*$.)

(b) Check that $v = \sum_i \mu_i(v) u_i$ is the unique representation of $v \in V$ in basis $B$.

(c) Show that the map $L : V \to F^n$ given by

$$
L(v) = \begin{pmatrix} 
\mu_1(v) \\
\vdots \\
\mu_n(v)
\end{pmatrix}
$$

is a linear isomorphism. (This means that $L$ is a bijective linear map. In other words, verify that the kernel, or null-space, of $L$ is $\{0\}$ and that $L$ is surjective.)

(d) Let $T : V \to V$ be a linear map, $B = \{u_1, \ldots, u_n\}$ a basis of $V$, and $B^* = \{\mu_1, \ldots, \mu_n\}$ the dual basis. Let $A$ be the matrix representing $T$ in basis $B$. (By definition, $A : F^n \to F^n$ is given by $A = LTL^{-1}$, where $L$ is as in the previous item.) Show that the $ij$-entry $a_{ij}$ of $A$ is $a_{ij} = \mu_i(Tu_j)$. (By definition $a_{ij}$ is the $i$th component of the column vector $Ae_j$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $F^n$.)

(e) Now suppose that $V$ is equipped with an inner product $\langle \cdot, \cdot \rangle$. (Generally, in this course, I will adopt the convention that inner products are linear in the second from left argument and conjugate-linear in the first.) Let $B = \{u_1, \ldots, u_n\}$ be an orthonormal basis of $V$. Show that $B^* = \{\mu_1, \ldots, \mu_n\}$ where $\mu_i(\cdot) = \langle u_i, \cdot \rangle$.

2. **The trace.** Let $V$ be a real or complex finite dimensional vector space. Let $T : V \to V$ be a linear map. The trace of $T$ is defined as follows: given a basis $B = \{u_1, \ldots, u_n\}$ of $V$ then

$$
\text{tr}(T) = \sum_{i=1}^n \mu_i(Tu_i)
$$
where $B^* = \{\mu_1, \ldots, \mu_n\}$ is the dual basis.

(a) Show that $\text{tr}(T)$ does not depend on the choice of basis.

(b) Recall the notion of direct sum: $V$ is the direct sum of subspaces $W_1, W_2$, denoted

$$V = W_1 \oplus W_2,$$

if $W_1 \cap W_2 = \{0\}$ and the union of the two subspaces linearly spans all of $V$. It follows that every $v \in V$ can be written in a unique way as $v = w_1 + w_2$ for $w_i \in W_i$. Now suppose that $V = W_1 \oplus W_2$ and that the linear map $T : V \to V$ maps $W_i$ into itself, $i = 1, 2$. Let us define $T_i : W_i \to W_i$ as the restriction of $T$ to $W_i$. Show that

$$\text{tr}(T) = \text{tr}(T_1) + \text{tr}(2).$$

(c) If $S : V \to V$ is another linear map, show that $\text{tr}(TS) = \text{tr}(ST)$. If $S$ is invertible, show that $\text{tr}(STS^{-1}) = \text{tr}(T)$.

(d) Let $A$ be the matrix that represents the linear map $T : V \to V$ in a given basis $B$. Show that $\text{tr}(T) = \text{tr}(A)$.

(e) Let $\lambda_1, \ldots, \lambda$ be the (generally complex) eigenvalues of $T$ counted with multiplicity. (So they are not necessarily distinct.) Show that the trace of $T$ equals the sum $\lambda_1 + \cdots + \lambda_n$. (Remark: according to Exercise 5 of Homework 1, for any square matrix $A$ it is possible to find a complex matrix $P$ such that $P^{-1}AP$ is upper triangular. Note that the eigenvalues of a triangular matrix are its diagonal entries, and that the eigenvalues of a linear transformation do not depend on the matrix representation we use to compute those eigenvalues. Convince yourself that this is true! You can prove it by considering the characteristic polynomials of matrices $A$ and $C^{-1}AC$. You need not write up a proof of this fact.)

3. The determinant. Recall that the symmetric group $S_n$ is the group of all permutations of $\{1, \ldots, n\}$, where a permutation is a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. The product and inverse of permutations are given by composition and inverse of functions. A special type of permutation is a transposition, denoted $(i j)$. This is the permutation that maps $i$ to $j$, $j$ to $i$, and all the other elements to themselves.

Every permutation $\sigma$ can be written as a product (not necessarily uniquely) of transpositions. In, fact recall from Homework Set 1 that every permutation can be written, essentially uniquely, as a product of disjoint cycles $(i_0 i_1 \ldots i_{k-1})$, and note that each cycle decomposes as a product of transpositions. (We regard the cycle as a function:

$$(i_0 i_1 \ldots i_{k-1}) : i_r \mapsto i_{r+1},$$

where the index is taken modulo $k$.) For example, let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}.$$ 

By this notation we mean that $\sigma$ maps $1 \mapsto 3$, $2 \mapsto 4$, etc. Then, in cycle notation, $\sigma = (135)(24)$. (Observe that the disjoint cycles commute with each other, so the order in which they are written does not matter.) Now notice that $(135) = (15)(13)$. So $\sigma = (15)(13)(24)$. More generally, convince yourself that $(i_0 i_1 \ldots i_{k-1}) = (i_0 i_{k-1}) \cdots (i_0 i_2)(i_0 i_1)$.

Now, the decomposition into transpositions is generally not unique, and the number of transpositions may also be different in two distinct factorizations of this kind. However, it can be proved that the parity of the
number of transpositions (that is, whether this number is even or odd) is uniquely determined. In other words, if a permutation is written as a product of an even number of transpositions, it is not possible to write it as a product of an odd number of other transpositions. With this in mind, we define the sign of $\sigma$ as

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is the product of an even number of transpositions} \\ -1 & \text{if } \sigma \text{ is the product of an odd number of transpositions}. \end{cases}$$

Thus, in the above example, $\text{sign}(\sigma) = -1$.

The determinant of $n \times n$ real or complex matrices may be defined as a function on the columns of the matrix as follows. Let $A_1, \ldots, A_n$ be the columns of $A$. Then $D(A_1, \ldots, A_n) = \det(A)$ satisfies:

- $A_i \in \mathbb{C}^n \mapsto D(A_1, \ldots, A_n) \in \mathbb{C}$ is a linear function for each $i$;
- $D(A_{\sigma(1)}, \ldots, A_{\sigma(n)}) = \text{sign}(\sigma)D(A_1, \ldots, A_n)$ for all permutations $\sigma \in S_n$;
- $D(e_1, \ldots, e_n) = 1$ where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. In other words, $D(I) = 1$.

It is not difficult to prove (you need not do it here) that these three properties uniquely determine $D$.

**Now to the questions.** (Do items (a), (b), and (c); simply read the others and try to convince yourself that the statements are correct, but you need not write down the proofs. You may use these facts in the last problem.)

(a) Find the decomposition into cycles and a decomposition into transpositions of $\sigma \in S_6$ defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 1 & 2 & 5 \end{pmatrix}.$$

What is $\text{sign}(\sigma)$?

(b) By only using the above three properties, show that

$$D\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc.$$

(c) Using that $D$ is uniquely characterized by the above three properties, show that

$$D(AB) = D(A)D(B)$$

for all matrices $A, B \in M_n(\mathbb{R})$. I suggest the following argument: First assume that $A$ is invertible and define

$$D_A(B) := D(AB_1, \ldots, AB_n)/D(A)$$

where $B_1, \ldots, B_n$ are the columns of $B$. Argue that $D_A$ satisfies the above three properties, so $D_A = D$. Now use the fact that a non-invertible matrix must have determinant 0 to conclude the argument. (Why is this so? Recall that a square matrix is not invertible if and only if its columns are linearly dependent vectors. Convince yourself that if one column of $A$ is a linear combination of the others, then $D(A) = 0$. No need to write this down.)

(d) Show that the determinant can be defined for a linear map $T : V \to V$ by noting that if $A$ and $B$ are any two matrix representations of $T$ associated to different choices of bases of $V$, then $A$ and $B$ must have the same determinant, which we define as the determinant of $T$. 

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(e) If $V$ is a finite dimensional complex vector space with inner product and $T: V \to V$ is a linear transformation, show that $\det(T^*) = \overline{\det(T)}$.

(f) If $V$ is a finite dimensional complex vector space with inner product and $U$ is a unitary transformation of $V$, show that the determinant of $U$ is a complex number of norm 1.

\[ \Box \]

4. **Examples of matrix groups.** Let the field $F$ be either $\mathbb{R}$ or $\mathbb{C}$. Denote by $GL(n, F)$ the space of all $n \times n$ matrices with entries in $F$. Then $GL(n, F)$ is a group with respect to matrix multiplication and inverse, called the general linear group in dimension $n$ and in the field $F$. A linear (or matrix) group is a subgroup of a general linear group. The groups one is often most interested in are linear groups whose entries satisfy additional polynomial equations. Let us look at at few examples here, taken from the text *Groups and Symmetries*, page 4.

- **The special linear group** over $F$:
  $SL(n, F) = \{ A \in GL(n, F) : \det A = 1 \}$

- **The orthogonal group** over $F$:
  $O(n, F) = \{ A \in GL(n, F) : A^t A = I \}$

- **The special orthogonal group** over $F$:
  $SO(n, F) = \{ A \in O(n, F) : \det A = 1 \}$

- For $n = p + q$, define the diagonal matrix $J_{pq} = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$. The pseudo-orthogonal group of type $(p, q)$:
  $O(p, q) = \{ A \in GL(n, \mathbb{R}) : AJ_{pq}A^t = J_{pq} \}$.

- **The special pseudo-orthogonal group** of type $(p, q)$:
  $SO(p, q) = \{ A \in O(p, q) : \det A = 1 \}$.

- **The unitary group**:
  $U(n) = \{ A \in GL(n, \mathbb{C}) : AA^* = I \}$

  where $A^* = \overline{A^t}$ is the adjoint, or conjugate-transpose, or $A$.

- **The special unitary group**:
  $SU(n) = \{ A \in U(n) : \det A = 1 \}$

For a complete list of the so-called classical groups, see https://en.wikipedia.org/wiki/Classical_group. Now the exercise:

(a) Show that $SO(p, q)$ is a subgroup of $GL(n, \mathbb{R})$.

(b) Show that $SU(n)$ is a subgroup of $GL(n, \mathbb{C})$.

Note: Showing that a subset $H$ of a group $G$ is a subgroup, requires checking that the product (relative to the multiplication in $G$) of any two elements in $H$ is in $H$, and the inverse of any element of $H$ is in $H$. 

\[ \Box \]