Homework set 7- due 11/21/30
Math 497 – Renato Feres

Some preliminaries

- **Invariant integration on** $G$. For this entire homework assignment, $G$ will denote a compact matrix group. It will be discussed in class that $G$ admits a unique measure which is both left- and right-invariant and with respect to which the measure of $G$ itself is 1. (I’ll also comment on the basic concepts from measure theory in class.) Invariance means that for every (measurable) set $U \subset G$ and all $A \in G$

  \[ \mu(AU) = \mu(UA) = \mu(U). \]

I will also use the term *probability measure* to indicate that $\mu$ is normalized: $\mu(G) = 1$. The integral of an integrable function $f : G \to \mathbb{C}$ is denoted

\[ \int_G f(A) \, d\mu(A) \]

or more simply $\int f \, d\mu$. Let $L_A, R_A : G \to G$ denote left- and right-translation by $A$:

\[ L_A(B) := AB, \quad R_A(B) := BA. \]

The invariance of $\mu$ can be equivalently expressed by

\[ \int_G f(AB) \, d\mu(A) = \int_G f(A) \, d\mu(A) = \int_G f(BA) \, d\mu(A) \]

for all integrable $f$ and all $B \in G$.

We use the existence of a (right-) invariant probability measure to obtain an invariant Hermitian inner product on any finite dimensional representation $(V, \Pi)$ of $G$: if $\langle \cdot, \cdot \rangle_0$ is an arbitrary Hermitian inner product on $V$, then by averaging with respect to $\mu$ we obtain an invariant inner product $\langle \cdot, \cdot \rangle$. That is, for all $u, v \in V$

\[ \langle u, v \rangle := \int_G \langle \Pi(A)u, \Pi(V)v \rangle_0 \, d\mu(A). \]

Invariance of the inner product amounts to

\[ \langle u, v \rangle = \langle \Pi(A)u, \Pi(A)v \rangle \]

for all $u, v \in V$ and all $A \in G$. In other words, with respect to the invariant inner product, the representation $\Pi$ on $V$ is unitary. It now follows from the existence of an invariant inner product that the representation is completely reducible. In fact, given any invariant subspace, its orthogonal complement with respect to $\langle \cdot, \cdot \rangle$ is also invariant.

- **$L^2$-spaces.** The main definitions here apply to much more general measure spaces than a compact matrix Lie
group $G$ with the invariant probability measure $\mu$. But for the sake of not introducing any more notation, I’ll restrict the discussion to this case.

Let $f$ be an integrable function on $G$ relative to the measure $\mu$. We define the vector space $L^2(G, \mu)$ to consist of all ($\mu$-measurable) complex-valued functions $f$ on $G$ such that

$$\|f\|_2^2 := \int_G |g|^2 \, d\mu < \infty.$$ 

Then $\|\cdot\|$ defines a norm on $L^2(G, \mu)$. This norm comes for the inner product

$$\langle f, g \rangle_{L^2} := \int_G f(A)\overline{g(A)} \, d\mu(A)$$

where $f, g$ are two elements in $L^2(G, \mu)$.

It can be shown that the topological vector space $L^2(G, \mu)$, with the metric induced by the $L^2$-norm is separable and complete. Complete means that any Cauchy sequence $f_1, f_2, \ldots$ of square integrable functions converges to a square integrable function in norm: $\|f_j - f\| \to 0$ as $k \to \infty$. A complete vector space with inner product is called a Hilbert space. Separable means that it admits a countable dense subset. It can be shown that a separable Hilbert space admits a countable orthonormal basis.

The notion of a basis of an infinite dimensional (separable) Hilbert space needs some elaboration. A subset $\{\varphi_1, \varphi_2, \ldots\} \subset L^2(G, \mu)$ is said to be an orthogonal family if

$$\langle \varphi_j, \varphi_k \rangle_{L^2} = \delta_{jk} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The orthogonal family is said to be complete if

$$\lim_{N \to \infty} \left\| f - \sum_{k=1}^N \langle f, \varphi_k \rangle \varphi_k \right\| = 0$$

for all $f \in L^2(G, \mu)$. Note that $\langle f, \varphi_k \rangle$ is the orthogonal projection of $f$ on the direction of the family element $\varphi_k$. This number is called the Fourier coefficient of $f$ relative to the orthogonal family. If the family is complete, we write

$$f = \sum_{k=1}^\infty \langle f, \varphi_k \rangle \varphi_k.$$ 

I will refer to a complete (countable) orthogonal family as an orthonormal basis of the (separable) Hilbert space.

A sufficient condition for the orthogonal family to be complete (for the compact group $G$), and thus to constitute an orthonormal basis, is that every continuous function on $G$ can be uniformly approximated, in the $L^2$-norm $\|\cdot\|$ by finite linear combinations of the $\varphi_k$. You may take also take the following simple fact for granted: If the Hilbert space is separable, then every orthogonal family in it must be countable (or finite). For general facts about Hilbert spaces, see for example A course in functional analysis by John B. Conway, Springer (Graduate Texts in Mathematics, V. 96, 1985).

- The Weierstrass approximation theorem. The following fundamental fact of analysis will be needed.

**Theorem 1** (Weierstrass’s approximation). Any continuous function on a compact subset of $\mathbb{R}^n$ may be uniformly approximated by polynomials.
This theorem will be needed below in the context of functions on a matrix group $G \subset GL(n, \mathbb{C})$. We say that a function $f : G \to \mathbb{C}$ is a polynomial (of degree $d$) if it is the restriction of a polynomial function on $\mathbb{R}^{2n}$, where we identify the latter space with the vector space of matrices $M(n, \mathbb{C})$. Thus $f$ is a function of the variables $x_{ij}$ such that $x_{ij}(A)$ is the $(i, j)$-entry of the matrix $A$.

Using basic facts about symmetric tensor spaces, we can describe the space of polynomials as follows. Let $V = \mathbb{C}^n$ be the vector space for the standard representation of the matrix group $G$ and $V^*$ the dual space. Recall that $V^*$ consists of linear functionals on $V$, which are first degree polynomials on $V$ with constant term equal to 0. A homogeneous polynomial of degree $m$ on $V$ is a linear combination of functions of the form

$$
\nu \mapsto \alpha(\nu) := \alpha_1(\nu) \cdots \alpha_m(\nu)
$$

for $\alpha_1, \ldots, \alpha_m \in V^*$. I will use the notation $V^{(m)}$ for the space of homogeneous polynomials on $V$ of degree $m$.

The space of polynomials on $V$ of degree $d$ is then the direct sum of the $V^{(m)}$ for $m \leq d$.

There are two natural ways of defining representations of $G$ on $V^{(m)}$ or on any space of functions on $G$:

$$(\Pi_l(A)f)(B) := f(A^{-1}B), \quad \text{and} \quad (\Pi_r(A)f)(B) := (BA).$$

Using the definition of the left and right-multiplications $L_A$ and $R_A$ defined at the beginning of this write-up, we have

$$\Pi_l(A)f := f \circ L_{A^{-1}} \quad \text{and} \quad \Pi_r(A)f := f \circ R_A.$$  

Check that both ways define representations of $G$. The subscripts “$l$” and “$r$” stand for “left” and “right.”

- **Schur’s lemma.** Given two representations $(V, \Pi)$ and $(W, \Lambda)$ of $G$ (not necessarily compact here), we say that a linear map $\varphi : V \to W$ is an intertwining map, or a $G$-map, if $\varphi \circ \Pi(A) = \Lambda(A) \circ \varphi$ for all $A \in G$.

I will use the notation $\Pi \approx \Lambda$ to mean that $\Pi$ and $\Lambda$ are equivalent representations, and $\Pi \not\approx \Lambda$ to mean that they are not. The following two properties have already been discussed in class. They amount to the main statement of Schur’s lemma: if $\varphi$ is an intertwining map for the two representations, then

1. $\Pi \not\approx \Lambda \implies \varphi = 0$;
2. $\Pi = \Lambda \implies \varphi = \lambda I$ for some $\lambda \in \mathbb{C}$.

If $G$ is a compact group, it can be shown that the statement holds for unitary representations on not necessarily finite dimensional Hilbert spaces. It can also be shown that the irreducible unitary representations of the compact group $G$ on a possibly infinite dimensional Hilbert space are finite dimensional.

- **Matrix coefficients of a representation.** Let $(V, \Pi)$ be a representation of $G$. Denote by $\Pi_{ij}(A)$ the $(i, j)$-coefficient (or entry) of the matrix associated to $\Pi(A)$ for a given basis of $V$. Let the basis be $\{u_1, \ldots, u_n\}$ and also consider the corresponding dual basis of the dual space $V^*$: $\{\alpha_1, \ldots, \alpha_n\}$. Recall that it is uniquely defined by the property

$$\alpha_j(u_j) = \delta_{ij}.$$  

Then observe:

$$\Pi_{ij}(A) = \alpha_i \left( \Pi(A) u_j \right).$$

More generally, if $u \in V$ and $\alpha \in V^*$ are arbitrary elements, we also call the function

$$A \in G \mapsto \alpha(\Pi(A)u) \in \mathbb{C}$$


a matrix coefficient of $\Pi$.

In one of this assignment’s problems you are going to show the following result.

**Theorem 2** (Schur’s orthogonality relations). Let $(V, \Pi)$ and $(W, \Lambda)$ be two irreducible unitary representations of the compact group $G$. Let $\Pi_{ij}(A)$ and $\Lambda_{kl}(A)$ be the matrix coefficients of $\Pi(A)$ and $\Lambda(A)$ with respect to orthonormal bases for $V$ and $W$. Then

$$\int_G \Pi_{ij}(A)\overline{\Lambda_{kl}(A)}\, d\mu(A) = \begin{cases} 0 & \text{if } \Pi \neq \Lambda \text{ or if } \Pi = \Lambda \text{ but } (i, j) \neq (k, l) \\ \frac{1}{\dim(\Pi)} & \text{if } \Pi = \Lambda \text{ and } (i, j) = (k, l). \end{cases}$$

- **The Peter-Weyl theorem.** Schur’s orthogonality relations imply that the set of matrix coefficients for the collection of pairwise inequivalent, finite dimensional, irreducible unitary representations of a compact matrix Lie group $G$ form an orthogonal family. The next theorem says that this family is complete.

**Theorem 3** (Peter-Weyl theorem for matrix Lie groups.). Let $G$ be a compact matrix Lie group. Let the set $\{\Pi^\nu\}_\nu$ consist of pairwise inequivalent finite-dimensional irreducible unitary representations of $G$ so that every irreducible finite dimensional unitary representation of $G$ is equivalent (i.e., isomorphic) to some $\Pi^\nu$. Then the collection of all matrix coefficients $\{\Pi^\nu_{ij}\}_{\lambda, i, j}$ forms a countable complete orthogonal family of functions on $G$. More precisely,

$$\langle \Pi^\lambda_{ij}, \Pi^\mu_{kl} \rangle = \begin{cases} 0 & \text{if } \Pi^\lambda_{ij} \neq \Pi^\mu_{kl} \\ \frac{1}{\dim(\Pi^\nu)} & \text{if } \Pi^\lambda_{ij} = \Pi^\mu_{kl} \end{cases}$$

and every $L^2$-function $f$ on $G$ admits an $L^2$-convergent Fourier expansion

$$f = \sum_{\lambda, i, j} \dim(\Pi^\lambda) \langle f, \Pi^\lambda_{ij} \rangle_{L^2} \Pi^\lambda_{ij}.$$
Problems

1. Preliminary. Read the preliminaries section. This is comparable in content to Chapter 3 (Representations of Compact Groups) of the text *Groups and Symmetries*, pages 33-42. Take a look, in particular, to section 2 (Haar measure).

2. The regular representation. Let $H := L^2(G, \mu)$ be the Hilbert space of all square-integrable functions on the compact matrix Lie group $G$. The left-regular representation of $G$ is the representation on $H$ given by the already defined $\Pi_l$. To recall, $\Pi_l(A) f := f \circ L_{A^{-1}}$ for all $f \in L^2(G, \mu)$. The right-regular representation of $G$ is similarly defined, now using $R_A$:

   $\Pi_r(A) f := f \circ R_A$.

(a) Check that $\Pi_l$ and $\Pi_r$ satisfy the homomorphism property $\Pi_l(AB) = \Pi_l(A) \Pi_l(B)$.
(b) Show that $\Pi_l$ and $\Pi_r$ are unitary representations.

Remark: The regular representations are continuous in the following sense: the map $A \mapsto \Pi_l(A) f$ is continuous in the norm topology of $H$ for each $f \in H$. You do not need to prove this here.

3. Orthogonality of matrix coefficients. Let $(V, \Pi)$ and $(W, \Lambda)$ be irreducible representations of the compact matrix group $G$ and $\psi : V \to W$ an arbitrary linear map (not necessarily intertwining). Define the linear map $\varphi : V \to W$ by averaging $\psi$ with respect to the invariant probability measure $\mu$ on $G$. That is, define

   $\varphi := \int_G \Lambda(A) \circ \psi \circ \Pi_l(A^{-1}) \, d\mu(A)$.

Let $\dim(\Pi)$ denote the dimension of the representation space $V$ of $\Pi$. Show the following:

(a) $\Pi \neq \Lambda \implies \varphi = 0$;
(b) $\Pi = \Lambda \implies \varphi = \lambda I$ where $\lambda = \text{tr}(\psi)/\dim(\Pi)$;
(c) Let $v \in V, a \in V^*, w \in W, \beta \in W^*$. Show that

   $\int_G a(\Pi_l(A)v) \beta(\Lambda(A^{-1})w) \, d\mu(A) = \begin{cases} 0 & \text{if } \Pi \neq \Lambda \\ \frac{1}{\dim(\Pi)} a(w)\beta(v) & \text{if } \Pi = \Lambda. \end{cases}$

(d) Suppose $\Pi$ and $\Lambda$ are irreducible, unitary representations. Show that for all $v_1, v_2 \in V$ and all $w_1, w_2 \in W$,

   $\int_G \langle \Pi_l(A)v_1, v_2 \rangle \langle \Lambda(A)w_1, w_2 \rangle \, d\mu(A) = \begin{cases} 0 & \text{if } \Pi \neq \Lambda \\ \frac{1}{\dim(\Pi)} \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle & \text{if } \Pi = \Lambda. \end{cases}$

(e) Conclude the proof of Schur’s orthogonality relations for matrix coefficients theorem. (This is stated above in the preliminaries section.)

4. Matrix coefficients. Let $W = V^{(m)}$ be the space of homogeneous polynomials of degree $m$ on the compact Lie group $G \subset GL(n, \mathbb{C})$. Let $\Pi$ be the representation $G$ on $W$ defined for each $f \in W$ by

   $(\Pi_l(A)f)(B) = f(BA)$.
(a) Show that $\delta_e : W \to C$ defined by $\delta_e(f) := f(e)$, where $e$ is the identity element of $G$, is an element of the dual space $W^*$.

(b) Show that each $f \in W$ is itself a matrix coefficient of the representation $\Pi$. Recall that a matrix coefficient was defined above as any function of $G$ of the form $c(A) = \alpha(\Pi(A)w)$ for $w \in W$ and $\alpha \in W^*$, for all $A \in G$. Suggestion: let $w = f$ and $\alpha = \delta_e$.

(c) Conclude that if $\langle \cdot, \cdot \rangle$ is an inner product on $W$, there must exist a homogenous polynomial $f_0$ in $W$ such that $f(A) = \langle f_0, \Pi(A)f \rangle$ for all $f \in W$.

The main remark here, that any function in $V^{(m)}$ is itself a matrix coefficient for some representation, will be needed shortly in the proof of Peter-Weyl's theorem.

5. The Peter-Weyl theorem for matrix groups. This exercise gives the proof of the Peter-Weyl theorem as stated in the preliminaries. First observe the following: Since $L^2(G, \mu)$ is a separable Hilbert space (see the preliminaries) the family

$$\left\{ \left[ \dim \left( \Pi^1 \right) \right]^{1/2} \Pi^1_{i,j} \right\}_{\lambda,i,j},$$

which is orthogonal by Schur's orthogonality theorem, must be countable. The Peter-Weyl theorem will then be established if we show that this orthogonal family is complete. As mentioned in the preliminaries, a sufficient condition for the orthogonal family to be complete is that continuous functions on $G$ can be uniformly approximated, in the $L^2$-norm, by (finite) linear combinations of elements in the family. By Weierstrass's approximation theorem, any continuous function on $G$ can be uniformly approximated by polynomial functions. Therefore the Peter-Weyl theorem (for compact matrix Lie groups) will be proved if we show that every polynomial function on $G$ is a linear combination of matrix coefficients of irreducible representations.

Thus, the problem is: Show that every polynomial function on $G$ can be written as a finite linear combination of elements in the above orthogonal family. Suggestion: see the remark at the end of the previous exercise.

6. Fourier series. In this problem we illustrate the Peter-Weyl in the special case of $T^n = \mathbb{R}^n / \mathbb{Z}^n$. Note that the $n$-dimensional torus can be expressed as a matrix group by identifying the torus with the group of unitary diagonal matrices

$$G = \{ z = \text{diag}(z_1, \ldots, z_n) : z_k \in \mathbb{C} \text{ and } |z_k| = 1 \text{ for } k = 1, \ldots, n \}. $$

The isomorphism is given by the map

$$t = [t_1, \ldots, t_n] \in \mathbb{R}^n / \mathbb{Z}^n \mapsto z = \text{diag}(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}) \in G.$$

(a) Explain: every irreducible unitary representation of the torus is one-dimensional and has the form

$$t \mapsto e^{2\pi i (l_1 t_1 + \cdots + l_n t_n)}$$

for some integer vector $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$. (You may take for granted, in the case of compact groups, what we have already shown for finite groups: every irreducible representation of an abelian group is 1-dimensional. The proof is actually the same. Also keep in mind Proposition 3.3, page 55 of the Groups and Symmetries text. It says that a continuous 1-parameter group $f(t)$ must have the form $f(t) = \exp(tX)$ where, for this problem, $X$ will be simply a real number.)

(b) Conclude that every $L^2$-function on $T^n$ (equivalently, a periodic $L^2$-function on the $n$-dimensional cube of
side length 1 in $\mathbb{R}^n$) can be expressed as a convergent Fourier series

$$f(t_1, \ldots, t_n) = \sum_{(l_1, \ldots, l_n) \in \mathbb{Z}^n} c_{l_1 \ldots l_n} e^{2\pi i (l_1 t_1 + \cdots + l_n t_n)}$$

where

$$c_{l_1 \ldots l_n} = \int_0^1 \cdots \int_0^1 f(t_1, \ldots, t_n) e^{-2\pi i (l_1 t_1 + \cdots + l_n t_n)} \, dt_1 \cdots dt_n.$$ 

Note: there is a unique invariant probability measure on the torus (equivalently, the unit cube with periodic boundary conditions), which is given by the standard volume measure $d\text{Vol}(t) = dt_1 \cdots dt_n$. 