Problem set 8 - due 12/13/18

Math 497 – Renato Feres

Problems

1. **Reading.** Read Chapter 5, pages 71 to 78 of the textbook.

2. **Exercise 5.4, page 79. Haar measure on SU(2).** We parametrize SU(2) off a set of measure zero by setting, for an element \(g \in SU(2), g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}\), \(a = a_1 + i a_2, b = b_1 + i b_2\), and

\[
a_1 = \cos \theta_3, \quad a_2 = \sin \theta_3 \cos \theta_2, \quad b_1 = \sin \theta_3 \sin \theta_2 \cos \theta_1, \quad b_2 = \sin \theta_3 \sin \theta_2 \sin \theta_1,
\]

for \(\theta_1 \in (0, 2\pi), \theta_2, \theta_3 \in (0, \pi)\). (Recall that SU(2) is topologically \(S^3\), the 3-dimensional sphere in 4-dimensional Euclidean space. Think about this parametrization geometrically, by taking “horizontal” 3-dimensional slices of \(S^3\), which are 2-dimensional spheres of radius \(\sin \theta_3\), then using standard spherical coordinates \(\theta_1, \theta_2\) on these sphere slices.) Of the following two items, write up the solution to part (a), and think about part (b):

(a) Show that the Haar integral on SU(2) can be written

\[
I(f) = \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \int_0^{2\pi} f(\theta_1, \theta_2, \theta_3) \sin^2 \theta_3 \sin \theta_2 d\theta_1 d\theta_2 d\theta_3.
\]

(In other words, show that the integral is normalized, so that \(I(1) = 1\), and that it is invariant under left (or right) translations by elements of SU(2). The second point requires doing a change of variables calculation.)

(b) Set \(\theta_3 = t\). Show that the Haar integral of a class function \(f\) is

\[
I(f) = \frac{2}{\pi} \int_0^\pi f(t) \sin^2 t \, dt.
\]

Recall that \(f\) is said to be a **class function** if it is invariant under conjugation: \(f(h) = f(g h g^{-1})\) for all \(g, h \in SU(2)\).

3. **Reading.** Read the statement of Exercise 5.5, page 80, for the definition of **Euler angles**.

4. **Exercise 5.6, page 80. Euler angles and Haar measure on SU(2).** Let \(\Omega = (0, 2\pi) \times (0, \pi) \times (-2\pi, 2\pi)\).

(a) Show that for every pair of complex numbers \(a\) and \(b\) such that \(|a|^2 + |b|^2 = 1\), \(\text{Im}(a) \neq 0\), and \(\text{Re}(b) \neq 0\), there is a unique triple \((\phi, \theta, \psi) \in \Omega\) such that

\[
a = \cos \frac{\theta}{2} e^{i(\phi + \psi)/2}, \quad b = i \sin \frac{\theta}{2} e^{i(\phi - \psi)/2}.
\]
(Note: I suspect it should be \( b = -\sin \frac{\theta}{2} e^{i(\phi - \psi)/2} \). You may use this value of \( b \) instead of the above.) Conclude that, off a set of measure zero of \( SU(2) \), each matrix \( g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in SU(2) \) can be written in one and only one way as

\[
g = \exp(\phi \xi_3) \exp(\theta \xi_2) \exp(\psi \xi_3)
\]

with \((\phi, \theta, \psi) \in \Omega\). (The \( \xi_i \) are defined on page 71.) Show that the image of \( g(\phi, \theta, \psi) \) under the morphism \( \varphi \) of \( SU(2) \) onto \( SO(3) \) is the rotation with Euler angles \((\phi, \theta, \psi)\).

(b) Show that the Haar integral on \( SU(2) \) can be written

\[
I(f) = \frac{1}{16\pi^2} \int_{\Omega} f(\phi, \theta, \psi) \sin \theta \, d\phi \, d\theta \, d\psi.
\]

Note: You may use the result of Exercise 5.5, page 80.

5. **Reading.** Read Chapter 6, pages 81 to 90.

6. **Exercise 6.2, page 91. The representations** \( D^\frac{1}{2} \) **and** \( D^1 \).

   (a) Write the matrices of \( J_3, J_+ \), and \( J_- \) in the basis \( f_m^l \) for the representations \( D^\frac{1}{2} \) and \( D^1 \).

   (b) Decompose \( D^\frac{1}{2} \otimes D^\frac{1}{2}, D^1 \otimes D^1, D^\frac{1}{2} \otimes D^1, \) and \( D^\frac{1}{2} \otimes D^\frac{1}{2} \otimes D^\frac{1}{2} \) into direct sums of irreducible representations.

7. **Reading.** Unfortunately, we may not have time to discuss Chapter 7. It deals with functions on the 2-sphere \( S^2 \), spherical harmonics, and the eigenvalue problem for the Laplace operator on \( S^2 \). Read the chapter to become acquainted with how these topics are connected to the representation theory of \( SO(3) \).