Homework set 1 - due 09/10/21

Math 5031

1. Read part 1 (on *Sets*) in prof. Kerr's notes, which are linked to in the course syllablus (`https://www.math.wustl.edu/~feres/Math5031Fall21/Math5031Fall21Syllabus.html`), with particular attention to the section on *Integers*.

2. Read part 2 (on *Groups*) up to, and including, *Groups and Subgroups*.

3. Let a, b be positive integers and $p_1, \ldots, p_s$ ($s \geq 1$) distinct primes such that $a = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and $b = p_1^{\beta_1} \cdots p_s^{\beta_s}$ where the α_i and β_i are non-negative integers. Show:

   (a) $\gcd(a,b) = \prod_{i=1}^{s} p_i^{\gamma_i}$ where $\gamma_i = \min\{\alpha_i, \beta_i\}$;

   (b) $\operatorname{lcm}(a,b) = \prod_{i=1}^{s} p_i^{\delta_i}$ where $\delta_i = \max\{\alpha_i, \beta_i\}$;

   (c) $\gcd(a,b)\operatorname{lcm}(a,b) = ab$;

   (d) If a (positive) prime p divides the product ab then p divides a or p divides b (or both).

   The greatest common divisor $\gcd(a,b)$ has already been defined. The largest common multiple $\operatorname{lcm}(a,b)$ is the unique (up to sign) integer m such that $a, b \mid m$ and if $a, b \mid \ell$ then $m \mid \ell$.

4. This problem concerns Example I.A.2 (iv) on page 6 of Kerr's notes. On $\mathbb{N}^2$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, we define a relation

$$(a,b) \sim (c,d) \iff a + d = b + c.$$

   Show:

   (a) $\sim$ is an equivalence relation;

   (b) There exists a bijective map between the quotient set $\mathbb{N}^2/\sim$ and the integers $\mathbb{Z}$.

5. Consider the permutation $\sigma = (456)(567)(671)(123)(234)(345) \in S_7$.

   (a) Write σ as a product of disjoint cycles;

   (b) Determine the sign of σ.

6. Show that if σ is any permutation in S_n, then $\sigma(i_1 i_2 \cdots i_r)\sigma^{-1} = (\sigma(i_1)\sigma(i_2)\cdots\sigma(i_r))$.

7. Show that any finite group of even order contains an element $a \neq 1$ such that $a^2 = 1$.

8. Show that a group G cannot be a union of two proper subgroups.

9. Let G be a group, not necessarily finite.

   (a) If every $a \in G$ satisfies $a^2 = 1$, show that G is abelian.

(b) Let G be the matrix group consisting of all matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where x, y, z belong to the cyclic (additive) group $\mathbb{Z}/3\mathbb{Z}$ of integers modulo 3. Show that this group is not abelian but all of its elements satisfy $a^3 = 1$.