Homework set 1 - due 09/10/21

Math 5031

1. Read part 1 (on Sets) in prof. Kerr’s notes, which are linked to in the course syllabus (https://www.math.wustl.edu/~feres/Math5031Fall21/Math5031Fall21Syllabus.html), with particular attention to the section on Integers.

2. Read part 2 (on Groups) up to, and including, Groups and Subgroups.

3. Let $a, b$ be positive integers and $p_1, \ldots, p_s$ ($s \geq 1$) distinct primes such that $a = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and $b = p_1^{\beta_1} \cdots p_s^{\beta_s}$ where the $\alpha_i$ and $\beta_i$ are non-negative integers. Show:
   (a) $\text{gcd}(a, b) = \prod_{i=1}^{s} p_i^{\gamma_i}$ where $\gamma_i = \min\{\alpha_i, \beta_i\}$;
   (b) $\text{lcm}(a, b) = \prod_{i=1}^{s} p_i^{\delta_i}$ where $\delta_i = \max\{\alpha_i, \beta_i\}$;
   (c) $\text{gcd}(a, b) \text{lcm}(a, b) = ab$;
   (d) If a (positive) prime $p$ divides the product $ab$ then $p$ divides $a$ or $p$ divides $b$ (or both).

   The greatest common divisor $\text{gcd}(a, b)$ has already been defined. The largest common multiple $\text{lcm}(a, b)$ is the unique (up to sign) integer $m$ such that $a, b | m$ and if $a, b | \ell$ then $m | \ell$.

4. This problem concerns Example I.A.2 (iv) on page 6 of Kerr's notes. On $\mathbb{N}^2$, where $\mathbb{N} = \{0, 1, 2, \ldots, \}$, we define a relation
   $$(a, b) \sim (c, d) \iff a + d = b + c.$$  
   Show:
   (a) $\sim$ is an equivalence relation;
   (b) There exists a bijective map between the quotient set $\mathbb{N}^2 / \sim$ and the integers $\mathbb{Z}$.

   (a) Write $\sigma$ as a product of disjoint cycles;
   (b) Determine the sign of $\sigma$.

6. Show that if $\sigma$ is any permutation in $S_n$, then $\sigma(i_1 i_2 \cdots i_r)\sigma^{-1} = (\sigma(i_1)\sigma(i_2)\cdots\sigma(i_r))$.

7. Show that any finite group of even order contains an element $a \neq 1$ such that $a^2 = 1$.

8. Show that a group $G$ cannot be a union of two proper subgroups.

9. Let $G$ be a group, not necessarily finite.
   (a) If every $a \in G$ satisfies $a^2 = 1$, show that $G$ is abelian.
(b) Let $G$ be the matrix group consisting of all matrices

$$
\begin{pmatrix}
1 & x & y \\
0 & 1 & z \\
0 & 0 & 1 \\
\end{pmatrix}
$$

where $x, y, z$ belong to the cyclic (additive) group $\mathbb{Z}/3\mathbb{Z}$ of integers modulo 3. Show that this group is not abelian but all of its elements satisfy $a^3 = 1$. 