Homework set 2 - due 09/17/21

Math 5031

Work out all the exercises listed below, but I'll ask you in Crowdmark to submit only exercises 2, 3, 4 and 5.

1. Read Sections II.D, II.E and II.F of Kerr's notes. (Also read the definitions and main statements from Section II.G up to Proposition II.G. They may be useful for parts (c), (d) and (e) of the below Exercise 2. I will likely discuss the proofs of results from that section after you worked on this exercise.)

2. Let G be a group, $a \in G$, and define the (conjugation by a) map $\varphi : G \to G$ as $\varphi(g) = aga^{-1}$.

   (a) Show that φ is an automorphism of G.

   (b) Let $G = S_n$ and $\alpha \in S_n$. Show that conjugation of cycles satisfies

   $$\alpha(i_1 i_2 \cdots i_r)\alpha^{-1} = (\alpha(i_1)\alpha(i_2)\cdots\alpha(i_r)).$$

   (c) Given a group G and an element $g_0 \in G$, let us define the set $\mathrm{ccl}_G(g_0) := \{gg_0g^{-1} : g \in G\}$. (Definition from Section II.G in Kerr's notes.) This set is called the *conjugacy class* of g_0 in G. What is the conjugacy class of the cycle $(1 \cdots r)$ in S_n?

   (d) How many elements does $\mathrm{ccl}_{S_n}((1 \cdots r))$ have? (You may use any theorem from Section II.G of Matt Kerr's notes.)

   (e) Describe the centralizer group of the cycle $(1 \cdots r)$. What is its order? (It may be helpful to find its order first. Keep in mind the Orbit-Stabilizer Theorem.)

3. Recall that if G is a finite group, then $\exp(G)$ is the least positive integer ℓ such that $g^\ell = 1$ for all $g \in G$.

   (a) Show that $\exp(S_n)$ is the least common multiple of $\{1, 2, \ldots, n\}$, where S_n is the symmetric group.

   (b) According to Corollary II.D.15 (of Proposition II.D.13) in Kerr's notes, if G is a finite group then G is cyclic if and only if $\exp(G) = |G|$ and G is abelian. If G is not assumed to be abelian, give a counterexample to the implication $\exp(G) = |G| \Longrightarrow G$ is cyclic. (Suggestion: try a symmetric group.)

   (c) Proposition II.D.13 states that if G is a finite abelian group then there exists $g \in G$ whose order is $\exp(G)$. Find a counterexample to this statement when G is nonabelian.

4. Show that the order of $\overline{a} = a + n\mathbb{Z}$ in $\mathbb{Z}_n$ is $n/(n, a)$, where (n, a) is the greatest common divisor of n and a.

5. Recall that $u \in \mathbb{Z}_n$ is a *unit* if there exists $v \in \mathbb{Z}_n$ such that $uv = \bar{1}$. (Here we are anticipating the fact that $\mathbb{Z}_n$ is a ring with the operations of addition and multiplication modulo n, and $\bar{1}$ is the multiplicative identity.) How many units does $\mathbb{Z}_{1000}$ have?

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6. The *dihedral group*, denoted D_n (sometimes D_{2n}) is the group of Euclidean symmetries of a regular n-gon. More precisely, identifying the coordinate plane $\mathbb{R}^2$ with $\mathbb{C}$, let us consider the n-gon with vertices at the nth roots of 1: $1, \omega, \omega^2, \omega^{n-1}$, where $\omega = e^{2\pi i/n}$. Let $r, s : \mathbb{C} \to \mathbb{C}$ be the maps defined by $r(z) = \omega z$ and $s(z) = \overline{z}$ (complex conjugation). The dihedral group D_n is then the subgroup of the group of orthogonal linear transformations of $\mathbb{R}^2$ generated by r and s. It is clear that r is an element of order n and s is an element of order 2.

   (a) Show that $(rs)^2 = 1$ and that $D_n = \{1, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\}$. In particular, $|D_n| = 2n$.

   (b) Show that D_3 is isomorphic with S_3.

   (c) Let $X = \{1, \ldots, n\}$ be the set of vertices of the n-gon. We obtain a homomorphism $\varphi : D_n \to S_n$ by associating to each $a \in D_n$ the permutation a defines on the vertices. Find the cycle representation of $\varphi(r)$ and $\varphi(s)$.

7. The *orthogonal group* in dimension n, denoted $O(n)$, consists of n-by-n matrices with real entries whose inverse equals the matrix transpose: $A^{-1} = A^{\top}$. The identity $A^{\top}A = I$ implies that the columns of A constitute an orthonormal basis of $\mathbb{R}^n$. Let S^{n-1} denote the sphere of unit radius centered at the origin of $\mathbb{R}^n$. So $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. Here $|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}$ (making use of the *dot product* from Calc III). We think of $x \in \mathbb{R}^n$ as a column vector. Recall that, for an arbitrary n-by-n real matrix A, the transpose of A satisfies the identity $(Ax) \cdot y = x \cdot (A^{\top}y)$ for all $x, y \in \mathbb{R}^n$. In particular, if A is orthogonal, $|Ax| = |x|$ for all $x \in \mathbb{R}^n$.

   (a) Define $O(n) \times \mathbb{R}^n \to \mathbb{R}^n$ by $(A, x) \mapsto Ax$. It is easy to check that this is a group action. What are its orbits?

   (b) Consider the action restricted from $\mathbb{R}^n$ to S^{n-1}. Let us write it as $\alpha : O(n) \times S^{n-1} \to S^{n-1}$. Show that α is a transitive action.

   (c) Let us refer to the standard basis vector $e = (0, \ldots, 0, 1)^{\top}$ as the *north pole* of the unit sphere. Describe the stabilizer subgroup $O(n)_e$.

   (d) Denote $G = O(n)$ and H the stabilizer of e. If $x \in S^{n-1}$, show that the stabilizer G_x is a conjugate of H in G.

8. The *special linear group* over the real numbers is the set of n-by-n real matrices having determinant 1. It is denoted $SL(n, \mathbb{R})$. For this exercise, you'll need to recall elementary properties of determinants, such as Cramer's rule and the formula for the determinant of a product of matrices. (We'll revisit determinants in greater generality later in the course.)

   (a) Show that $SL(n, \mathbb{R})$ is indeed a group. (It is a subgroup of the general linear group, $GL(n, \mathbb{R})$, of all invertible n-by-n real matrices.)

   (b) Show that $SL(n, \mathbb{Z})$, defined as the subset of $SL(n, \mathbb{R})$ consisting of matrices with integer entries, is a subgroup of $SL(n, \mathbb{R})$.

9. Define the 4-by-4 matrices I, J, K:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathscr{I} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathscr{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathscr{K} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

   (a) Show that $Q = \{\pm I, \pm\mathscr{I} \pm \mathscr{J}, \pm\mathscr{K}\}$ is a finite subgroup, Q, of the orthogonal group $O(4)$.

   (b) Show that Q is isomorphic to the *quaternion group* introduced in Definition II.G.22 of Kerr's notes.