Homework set 5 - due 10/08/21

Math 5031

Work on all the exercises, but turn in only numbers 2, 3, 4 and 5.

1. Show that if p is a positive prime, then a group of order p^2 is abelian and isomorphic to either $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.
2. For this exercise, review Section II.L of Kerr's notes on the Sylow theorems.
   (a) Explain: 3-Sylow subgroups of S_6 are abelian.
   (b) Show that the 3-Sylow subgroups of S_6 are isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.
   (c) Show that subgroups of S_6 isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ are generated by two disjoint 3-cycles.
   (d) Find all Sylow 3-subgroups of S_6. How many are there?
   (e) Check that the number of Sylow 3-subgroups you obtained in the previous item is compatible with the constraints imposed by the Sylow theorems.
3. Classify up to isomorphism the groups of order 245.
4. For this exercise, read the main definitions and results in Section II.J of Kerr's notes on automorphisms of groups. Here's a quick review of the main ideas. An *automorphism* of a group G is an isomorphism $\varphi : G \to G$. The collection of all automorphisms (with the operations of composition and map inverse) constitutes the group of automorphisms of G, denoted $\mathrm{Aut}(G)$. The *inner* automorphisms of G are those elements of $\mathrm{Aut}(G)$ given by conjugation: $h \mapsto i_g(h) = ghg^{-1}$. The inner automorphisms form a subgroup $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$. It is not difficult to prove (see Proposition-Definition II.J.3) that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. The quotient group $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the group of *outer* automorphisms of G. If G is abelian, $\mathrm{Inn}(G)$ is trivial and $\mathrm{Out}(G) = \mathrm{Aut}(G)$.

   In Examples II.J.4, page 71, it is shown that $\mathrm{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. (Recall the group $\mathbb{Z}_n^*$ from Section II.C, Example II.E.13 and Proposition II.E.14.) Let us now consider the automorphism group of a direct product of abelian groups.
   (a) Suppose m, n are relatively prime positive integers. Show that $\mathrm{Aut}(\mathbb{Z}_m \times \mathbb{Z}_n) \cong \mathbb{Z}_m^* \times \mathbb{Z}_n^*$.
   (b) Now suppose $m = n = p$ is a prime. Show that $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p) \cong \mathrm{GL}(2, \mathbb{Z}_p)$, where the latter is the group of 2-by-2 invertible matrices with entries in $\mathbb{Z}_p$. (Think of $\mathbb{Z}_p \times \mathbb{Z}_p$ as a "vector space" with coefficients in $\mathbb{Z}$; in fact, we may regard elements of this product group as column vectors of dimension 2.)
   (c) Find the order of $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p)$. (By the previous item, this reduces to counting the number of invertible matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients in $\mathbb{Z}_p$. Play with the determinant!)
5. This exercise has to do with the notion of *semidirect product* of groups, given in Definition II.M.15, page 94, of Kerr's notes. Here's the definition: Let $\theta : H \to \mathrm{Aut}(K)$ be a homomorphism sending $h \mapsto \theta_h$. The *semidirect product* $K \rtimes_\theta H$ is the group with underlying set $K \times H$ and product

$$(k, h)\left(k', h'\right) := \left(k\theta_h\left(k'\right), hh'\right).$$

The identity element is $(1_K, 1_H)$ and $(k,h)^{-1} = \left(\theta_{h^{-1}}\left(k^{-1}\right), h^{-1}\right)$. (Check that these are, in fact, the identity and inverse. You don't need to write this down.)

(a) Write explicitly the result conjugating $(k,h)\left(k',h'\right)(k,h)^{-1}$ in the semidirect product. In particular, show that $K \times \{1_H\}$ is a normal subgroup of $K \rtimes_\theta H$. Based on this fact, convince yourself (no need to write it down!) that the notation for semidirect product is reasonable.

(b) Let $O(n)$ denote the orthogonal group of $\mathbb{R}^n$. The *Euclidean group* in dimension n is the group of transformations of $\mathbb{R}^n$ generated by orthogonal transformations and translations: $Tx := Ax + a$, where $A \in O(n)$, $a \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$. Writing elements of the Euclidean group as pairs (a, A), show that this group can be expressed as a semidirect product of the groups $\mathbb{R}^n$ (the abelian group of vectors under addition) and $O(n)$. What is the homomorphism θ?

(c) Show that, up to isomorphism, groups of order 490 are semidirect products $K \rtimes \mathbb{Z}_2$ where K is a group of order 245 (already classified in Exercise 3. What is θ? (It is not unique; describe it in general terms.)

6. The *Heisenberg group mod* 3 is the matrix group (under matrix multiplication and matrix inverse for the group operations) of order 27 defined as

$$\mathcal{H} := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \text{ with } a,b,c \in \mathbb{Z}_3 \right\}.$$

Let

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and let I be the identity matrix. In this exercise, we wish to find all the subgroups of the mod 3 Heisenberg group.

(a) Show that x, y, z generate G and satisfy the relations

$$x^3 = y^3 = z^3 = I, \;\; [x,z] = [y,z] = I, \;\; z = [x,y] := x^{-1}y^{-1}xy.$$

Also show that $[\mathcal{H},\mathcal{H}] = \langle z \rangle = C(\mathcal{H})$ where the latter is the center of $\mathcal{H}$.

(b) Show that $\langle z \rangle$ is the unique normal subgroup of $\mathcal{H}$ of order 3 and that $\mathcal{H}/[\mathcal{H},\mathcal{H}] \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

(c) Show that any subgroup of $\mathcal{H}$ of order 9 contains $[\mathcal{H},\mathcal{H}]$ and is normal in $\mathcal{H}$. How many are there?

(d) How many subgroups of order 3 are there in $\mathcal{H}$? (Check that all the elements of $\mathcal{H} \setminus \{1\}$ have order 3.)