Homework set 6 - due 10/22/21
Math 5031

Solve all the exercises but turn in only numbers 1, 2, 3, 5, 7.

1. Let $R$ be a ring. An element $e \in R$ is said to be idempotent if $e^2 = e$; $z \in R$ is said to be nilpotent if $z^n = 0$ for some positive integer $n$. An element $u$ of $R$ has a right inverse if there exists $u' \in R$ such that $uu' = 1$. It has a left inverse $u''$ if $u''u = 1$. We say that $u \in R$ is a unit (or invertible element) if there is a $u' \in R$ which is both a right and left inverse of $u$. The ring $R$ is a domain if it contains no zero-divisors. Show the following.

(a) A domain contains no idempotents except $e = 0$ and $e = 1$.
(b) The only nilpotent element in a domain is $z = 0$.
(c) Let $u$ be a ring element that has a right inverse. Show that the following are equivalent:
   
   i. $u$ has more than one right inverse;
   ii. $u$ is not a unit;
   iii. $u$ is a left zero-divisor.

(d) (This part is not required. Try it as a challenge.) If $u$ is an element of a ring that has more than one right inverse, then $u$ has infinitely many. [Suggestion: let $s \neq 0$ be such that $us = 0$. Show that $su^n$ are all distinct for $n \in \mathbb{N}$.]

2. Using elementary row operations, show that

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \\ -3 & -6 & -8 \end{pmatrix}$$

is invertible in $M_3(\mathbb{Z})$ and find its inverse.

3. Prove that if $R$ is a commutative ring then $AB = \mathbb{I}$ in $M_n(R)$ implies $BA = \mathbb{I}$. (Suggestion: use the determinant.)

4. Let $R$ be a ring. Determine the set of matrices in $M_n(R)$ that commute with $N = e_{12} + e_{23} + \cdots + e_{n-1,n}$.

5. Read the item on number rings at the beginning of Chapter III (on rings). Let $d$ be a positive integer. Find the group of units in the ring $R$ where

(a) $R = \mathbb{Z} \left[ \sqrt{-d} \right]$.

(b) $R = \mathbb{Z} \left[ \frac{1+\sqrt{-d}}{2} \right]$ where $d \equiv 3(\text{mod } 4)$.

6. Recall that a domain $R$ is said to be an Euclidean domain if there is a function $\delta : R \setminus \{0\} \to \{1,2,\ldots\}$ for which the following holds: whenever $a,b \in R$ with $b \neq 0$, there are $q,r \in R$ with

$$a = qb + r$$
and either $r = 0$ or $\delta(r) < \delta(b)$. Such $\delta$ is called an *Euclidean function* for $R$.

Let us consider

$$R = \mathbb{Z} \left[ \frac{1 + \sqrt{-d}}{2} \right] = \left\{ m + n \frac{1 + \sqrt{-d}}{2} : m, n \in \mathbb{Z} \right\} = \left\{ \frac{a + b\sqrt{-d}}{2} : a, b \in \mathbb{Z}, a \equiv b \pmod{2} \right\}$$

where $d$ is square free and $d \equiv 3 \pmod{4}$. (See the initial pages of Chapter 3 concerning number rings.) Let $\mathcal{N}(r) = r \bar{r}$ where $\bar{r}$ is the complex conjugate of $r$:

$$\frac{a + b\sqrt{-d}}{2} = \frac{a - b\sqrt{-d}}{2}.$$

So $\mathcal{N}(r) = |r|^2$ is the square of the ordinary Euclidean norm of the complex number $r$.

(a) Suppose that for any complex number $z$ there is $\xi \in R$ such that $\mathcal{N}(z - \xi) < 1$. Show that $\mathcal{N}$ is an Euclidean function on $R$, making $R$ an Euclidean domain.

(b) Show that $R = \mathbb{Z} \left[ \frac{1 + \sqrt{-3}}{2} \right]$ is an Euclidean domain.

7. Show that $\pm 3$ are not of the form $m^2 - 10n^2$, $m, n \in \mathbb{Z}$. (Suggestion: reduce modulo 5.)