Homework set 12
Math 5031

1. (Jacobson, page 188 exercise 1.) Determine the structure of \( \mathbb{Z}^3/K \), where \( K \) is generated by \( f_1 = (2, 1, -3), f_2 = (1, -1, 2) \).

2. Let \( G \) be the abelian group determined by generators \( V, W, X, Y, Z \) and relations

\[
\begin{align*}
V - 7W + 14Y - 21Z &= 0 \\
15V - 7W - 2X + 10Y - 5Z &= 0 \\
7V - 3W - 2X + 6Y - 9Z &= 0 \\
V - 3W + 2Y - 9Z &= 0.
\end{align*}
\]

Find the rank and torsion invariants (invariant factors) by putting an appropriate matrix in normal form. Write \( G \) in the form

\[
\mathbb{Z}^{d_1} \oplus \cdots \oplus \mathbb{Z}^{d_k} \oplus \mathbb{Z}^r.
\]

Here \( G/G_{\text{tor}} = \mathbb{Z}^r \) and \( r \) is the rank of \( G \).

3. (Jacobson, page 181 exercise 2.) Find a base for the submodule \( K \) of \( \mathbb{Q}[\lambda]^3 \) generated by

\[
f_1 = (2\lambda - 1, \lambda, \lambda^2 + 3), \ f_2 = (\lambda, \lambda, \lambda^2), \ f_3 = (\lambda + 1, 2\lambda, 2\lambda^2 - 3).
\]

4. Find a normal form for the matrix

\[
A = \begin{pmatrix}
\lambda^2 - 3\lambda + 2 & \lambda - 2 \\
(\lambda - 1)^3 & \lambda^2 - 3\lambda + 2
\end{pmatrix}
\]

in \( M_2(\mathbb{Q}[\lambda]) \), \( \lambda \) an indeterminate. Also find invertible matrices \( P \) and \( Q \) such that \( PAQ \) is in normal form.

5. (Jacobson, page 186 exercise 3.) Determine the invariant factors of

\[
A = \begin{pmatrix}
\lambda + 1 & 2 & -6 \\
1 & \lambda & -3 \\
1 & 1 & \lambda - 4
\end{pmatrix}
\]

both by putting it in normal form (to get the \( d_i \) directly), and by using \( d_1 = \Delta_1, d_2 = \frac{\Delta_2}{\Delta_1}, d_3 = \frac{\Delta_3}{\Delta_2} \) (without putting \( A \) in normal form.)

6. Prove the Cayley-Hamilton Theorem: If \( B \in M_n(\mathbb{F}) \) and \( p_B(\lambda) := \det(\lambda \mathbb{1} - B) \) is the characteristic polynomial of \( B \), then \( p_B(B) = 0 \).
7. In Exercise 5, \( A = \lambda \mathbb{1} - B \) for \( B \in M_3(\mathbb{Q}) \). Determine the minimal polynomial of \( B \) as well as its rational and Jordan canonical forms.

8. (Jacobson, page 186 exercise 10.) Let \( R \) be a (not necessarily commutative) ring and define the elementary matrix \( T_{ij}(a) := T_{ij}^{(a)}(a) := \mathbb{1}_{n + a} e_{ij}, i \neq j \). Here \( a \in R \) and \( e_{ij} \) is the matrix having 1 at the \((i, j)\)-entry and zeros everywhere else. Verify the four Steinberg relations:

(a) \((T_{ij}(a))^{-1} = T_{ij}(-a)\);

(b) \( T_{ij}(a) T_{ij}(b) = T_{ij}(a + b) \);

(c) \([T_{ij}(a), T_{jk}(b)] = T_{ik}(ab)\) if \( k \neq i \) (where the commutator is \([x, y] := x^{-1} y^{-1} xy\));

(d) \([T_{ij}(a), T_{k\ell}(b)] = 1\) if \( j \neq k \) and \( i \neq \ell \).

9. (Jacobson, page 202 exercise 8.) Prove that any nilpotent matrix in \( M_n(\mathbb{F}) \) is similar to

\[
\begin{pmatrix}
N_1 & 0 \\
& \ddots \\
0 & N_k
\end{pmatrix}
\]

where the \( N_i \) are blocks of the form

\[
\begin{pmatrix}
0 & 1 \\
& \ddots \\
0 & 1
\end{pmatrix}
\]

10. (Jacobson, page 202 exercise 4.) Verify that the characteristic polynomial of

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-2 & -2 & 0 & 1 \\
-2 & 0 & -1 & -2
\end{pmatrix}
\]

is a product of linear factors in \( \mathbb{Q}[\lambda] \). Determine the rational and Jordan canonical forms for \( A \) in \( M_4(\mathbb{Q}) \). Also find a matrix \( S \) such that \( SAS^{-1} \) equals the Jordan canonical form.

11. (Jacobson, page 193 exercise 1.) Let \( R = \mathbb{R}[\lambda] \) and suppose \( M \) is a direct sum of cyclic \( R \)-modules of the following form:

\[
M = R/(\lambda - 1)^3 \oplus R/(\lambda^2 + 1)^2 \oplus R/(\lambda - 1)(\lambda^2 + 1)^4 \oplus R/(\lambda + 2)(\lambda^2 + 1)^2.
\]

Determine the elementary divisors and invariant factors of \( M \).

12. (Jacobson, page 188 exercise 2.) Determine the structure of \( M = \mathbb{Z}[i]^3 / K \) where \( K \) is generated by

\[
f_1 = (1, 3, 6), \ f_2 = (2 + 3i, -3i, 12 - 18i), \ f_3 = (2 - 3i, 6 + 9i, -18i).
\]

(This is similar to the above Exercise 2 in that you need to write down a matrix \( A \) and reduce it to normal form. Note that \( \mathbb{Z}[i] \) is Euclidean; you can simply use the absolute value.)

13. Determine the number of non-isomorphic abelian groups of order 900, and find the invariant factors \( d_i \) (in the structure theorem) for each group. (Hint: use IV.C.13-14, then convert to the form IV.C.10.)