The below problems are all from Tu’s book.

1. Read sections §1 and §2 of Chapter 1 of Tu’s book.

2. Tu, Problem 1.2, page 8. A $C^\infty$ function very flat at 0. Let $f(x)$ be the function on $\mathbb{R}$ defined in Example 1.3.

   (a) Show by induction that for $x > 0$ and $k \geq 0$, the $k$th derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in $y$.

   (b) Prove that $f$ is $C^\infty$ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.

3. Problem 1.5, page 8. A diffeomorphism of an open ball with $\mathbb{R}^n$. Let 0 = (0, 0) be the origin and $B(0, 1)$ the open unit disk in $\mathbb{R}^3$. To find a diffeomorphism between $B(0, 1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ with the $xy$-plane in $\mathbb{R}^3$ and introduce the lower open hemisphere

   \[ S : x^2 + y^2 + (z-1)^2 = 1, \quad z < 1, \]

   in $\mathbb{R}^3$ as an intermediate space (Figure 1.4, page 9 in text.) First note that the map

   \[ f : B(0,1) \to S, \quad (a, b) \mapsto (a, b, 1 - \sqrt{1 - a^2 - b^2}) \]

   is a bijection.

   (a) The stereographic projection $g : S \to \mathbb{R}^2$ from (0, 0, 1) is the map that sends a point $(a, b, c) \in S$ to the intersection of the line through (0, 0, 1) and $(a, b, c)$ with the $xy$-plane. Show that it is given by

   \[ (a, b, c) \mapsto (u, v) = \left( \frac{a}{1 - c}, \frac{b}{1 - c}, 1 - \sqrt{1 - a^2 - b^2} \right), \quad c = 1 - \sqrt{1 - a^2 - b^2}, \]

   with inverse

   \[ (u, v) \mapsto \left( \frac{u}{\sqrt{1 + u^2 + v^2}}, \frac{v}{\sqrt{1 + u^2 + v^2}}, 1 - \frac{1}{\sqrt{1 + u^2 + v^2}} \right). \]

   (b) Composing the two maps $f$ and $g$ gives the map

   \[ h = g \circ f : B(0,1) \to \mathbb{R}^2, \quad h(a,b) = \left( \frac{a}{\sqrt{1 - a^2 - b^2}}, \frac{b}{\sqrt{1 - a^2 - b^2}} \right). \]

   (c) Generalize part (b) to $\mathbb{R}^n$.

4. Problem 2.1, page 17. Vector fields. Let $X$ be the vector field $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ and $f(x, y, z)$ the function $x^2 + y^2 + z^2$ on $\mathbb{R}^3$. Compute $Xf$. 


5. **Problem 2.2, page 17. Algebra structure on** $C_p^\infty$. Define carefully addition, multiplication, and scalar multiplication in $C_p^\infty$. Prove that addition in $C_p^\infty$ is commutative.

6. **Problem 2.3, page 17. Vector space structure on derivations at a point.** Let $D$ and $D'$ be derivations at $p$ in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that
   
   (a) the sum $D + D'$ is a derivation at $p$.
   (b) the scalar multiple $cD$ is a derivation at $p$.

7. **Problem 2.4, page 17. Product of derivations.** Let $A$ be an algebra over a field $K$. If $D_1$ and $D_2$ are derivations of $A$, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if $D_1$ or $D_2$ is 0), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of $A$. 