Homework set 3 - Due 9/21/18

Math 5041 – Renato Feres

1. Tu’s book, pages 18-32. Read Chapter 1, Section 3.

2. Problem 3.1, page 32, Tu. Tensor product of covectors. Let $e_1, \ldots, e_n$ be a basis for a vector space $V$ and let $\alpha^1, \ldots, \alpha^n$ be its dual basis in $V^*$. (Note that I’m using a different notation for the dual space than Tu.) Suppose $(g_{ij})$ is an $n \times n$ real matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{i,j} g_{ij} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum j w^j e_j$. Describe $f$ in terms of the tensor products of $\alpha^i \otimes \alpha^j$.

3. Problem 3.3, page 32, Tu. A basis for $k$-tensors. Let $V$ be a vector space of dimension $n$ with basis $e_1, \ldots, e_n$. Let $\alpha^1, \ldots, \alpha^n$ be the dual basis for $V^*$. Show that a basis for the space $L_k(V)$ of $k$-linear functions on $V$ is the set of $\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}$ for all multi-indices $(i_1, \ldots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(V)$). In particular, this shows that $\dim L_k(V) = n^k$.

4. Problem 3.8, page 33, Tu. Transformation rule for $k$-covectors. Let $f$ be a $k$-covector on a vector space $V$. Suppose two sets of vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ are related by

$$u_j = \sum_{i=1}^k a_{ij} v_i, \quad j = 1, \ldots, k,$$

for a $k \times k$ matrix $A = (a_{ij})$. Show that

$$f(u_1, \ldots, u_k) = (\det A) f(v_1, \ldots, v_k).$$

5. Problem 3.10, page 33, Tu. Linear independence of covectors. Let $\alpha^1, \ldots, \alpha^k$ be 1-covectors on a vector space $V$. Show that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space $V^*$.

6. Problem 3.11, page 33, Tu. Exterior multiplication. Let $\alpha$ be a nonzero 1-covector and $\gamma$ a $k$-covector on a finite-dimensional vector space $V$. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k - 1)$-covector $\beta$ on $V$.

7. Tu's book, page 34-40. Read Chapter 1, Sections 4.1, 4.2, and 4.3.

8. Problem 4.2, page 44, Tu. A 2-form on $\mathbb{R}^3$. At each point $p \in \mathbb{R}^3$, define a bilinear function $\omega_p$ on $T_p \mathbb{R}^3$ by

$$\omega_p(a, b) = \left( \frac{a^1}{a^2} \right) \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} \cdot \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} = \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \\ a^3 & b^3 \end{bmatrix}.$$
for tangent vectors $a, b \in T_p\mathbb{R}^3$, where $p^3$ is the third component of $p = (p^1, p^2, p^3)$. Since $\omega_p$ is an alternating bilinear function on $T_p\mathbb{R}^3$, $\omega$ is a 2-form on $\mathbb{R}^3$. Write $\omega$ in terms of the standard basis $dx^i \wedge dx^j$ at each point.

9. **Problem 4.4, page 45, Tu. Spherical coordinates.** Suppose the standard coordinates on $\mathbb{R}^3$ are called $\rho, \phi$ and $\theta$. If

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

calculate $dx, dy, dz$ and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

10. **Problem 4.5, page 45, Tu. Wedge product.** Let $\alpha$ be a 1-form and $\beta$ a 2-form on $\mathbb{R}^3$. Then

$$\alpha = a_1 \, dx^1 + a_2 \, dx^2 + a_3 \, dx^3,$$

$$\beta = b_1 \, dx^2 \wedge dx^3 + b_2 \, dx^3 \wedge dx^1 + b_3 \, dx^1 \wedge dx^2.$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.