Homework set 3 - Solutions
Math 5041 – Renato Feres

1. Tu’s book, pages 18-32. Read Chapter 1, Section 3.

2. Problem 3.1, page 32, Tu. Tensor product of covectors. Let $e_1, \ldots, e_n$ be a basis for a vector space $V$ and let $\alpha^1, \ldots, \alpha^n$ be its dual basis in $V^*$. (Note that I’m using a different notation for the dual space than Tu.) Suppose $(g_{ij})$ is an $n \times n$ real matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{i,j} g_{ij} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$. Describe $f$ in terms of the tensor products of $\alpha^i \otimes \alpha^j$. This is easily seen by evaluating the right-hand side on $v, w$:

**Solution.** I claim that $f = \sum_{i,j} g_{ij} \alpha^i \otimes \alpha^j$.

If fact,

$$\left( \sum_{i,j} g_{ij} \alpha^i \otimes \alpha^j \right)(v, w) = \sum_{i,j} g_{ij} \alpha^i(v) \alpha^j(w) = \sum_{i,j} g_{ij} v^i w^j = f(v, w).$$

3. Problem 3.3, page 32, Tu. A basis for $k$-tensors. Let $V$ be a vector space of dimension $n$ with basis $e_1, \ldots, e_n$. Let $\alpha^1, \ldots, \alpha^n$ be the dual basis for $V^*$. Show that a basis for the space $L_k(V)$ of $k$-linear functions on $V$ is the set of $\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}$ for all multi-indices $(i_1, \ldots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(V)$). In particular, this shows that $\dim L_k(V) = n^k$.

**Solution.** Let us first check that the set of $\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}$ constitute a basis for $L_k(V)$. Observe that

$$\left( \sum_{i_1, \ldots, i_k} a_{i_1 \ldots i_k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k} \right)(e_{j_1}, \ldots, e_{j_k}) = \sum_{i_1, \ldots, i_k} a_{i_1 \ldots i_k} \alpha^{i_1}(e_{j_1}) \cdots \alpha^{i_k}(e_{j_k}) = a_{j_1 \ldots j_k}.$$
To show the set linearly spans $L_k(V)$, let $f$ be an arbitrary $k$-tensor in $L_k(V)$ and consider the difference

$$h = f - \sum_{i_1, \ldots, i_k} f(e_{i_1}, \ldots, e_{i_k}) a^{i_1} \otimes \cdots \otimes a^{i_k}.$$  

Then

$$h(e_{j_1}, \ldots, e_{j_k}) = f(e_{j_1}, \ldots, e_{j_k}) - \sum_{i_1, \ldots, i_k} f(e_{i_1}, \ldots, e_{i_k}) a^{i_1}(e_{j_1}) \cdots a^{i_k}(e_{j_k}) = \sum_{i_1, \ldots, i_k} a^{i_1}(e_{j_1}) \cdots a^{i_k}(e_{j_k}) = 0.$$  

Since this is true for all multi-indices $(j_1, \ldots, j_k)$ we conclude that the multilinear function $h$ is 0. Therefore $f$ can be written as a linear combination of the $a^{i_1} \otimes \cdots \otimes a^{i_k}$.

\[ \diamond \]

4. **Problem 3.8, page 33, Tu. Transformation rule for $k$-covectors.** Let $f$ be a $k$-covector on a vector space $V$. Suppose two sets of vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ are related by

$$u_j = \sum_{i=1}^k a^j_i v_i, \quad j = 1, \ldots, k,$$

for a $k \times k$ matrix $A = \{a^j_i\}$. Show that

$$f(u_1, \ldots, u_k) = (\det A) f(v_1, \ldots, v_k).$$

**Solution.** By definition, a $k$-covector is an alternating $k$-form. Thus

$$f(u_1, \ldots, u_k) = f\left(\sum_{j_1} a^j_{i_1} v_{j_1}, \ldots, \sum_{j_k} a^j_{i_k} v_{j_k}\right) = \sum_{j_1, \ldots, j_k} a^j_{i_1} \cdots a^j_{i_k} f(v_{j_1}, \ldots, v_{j_k}).$$

Since $f$ is alternating, $f(v_{j_1}, \ldots, v_{j_k}) = 0$ whenever two subindices are equal. Thus we may only take into account those multi-indices that are permutations of $\{1, \ldots, k\}$. This implies

$$f(u_1, \ldots, u_k) = \sum_{\sigma \in S_k} a^{(1)}_1 \cdots a^{(k)}_k f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \sum_{\sigma \in S_k} \text{sign}(\sigma) a^{(1)}_1 \cdots a^{(k)}_k f(v_1, \ldots, v_k) = (\det A) f(v_1, \ldots, v_k).$$

\[ \diamond \]

5. **Problem 3.10, page 33, Tu. Linear independence of covectors.** Let $a^1, \ldots, a^k$ be 1-covectors on a vector space $V$. Show that $a^1 \wedge \cdots \wedge a^k \neq 0$ if and only if $a^1, \ldots, a^k$ are linearly independent in the dual space $V^*$.  

**Solution.** Suppose the 1-covectors are linearly dependent. Then for some index $i$, we may write $a^i = \sum_{j \neq i} a_j a^j$. Therefore

$$a^1 \wedge \cdots \wedge a^k = a^1 \wedge \cdots \wedge a^i \wedge \cdots \wedge a^k = \sum_{j \neq i} a_j a^1 \wedge \cdots \wedge a^{i-1} \wedge a^j \wedge a^{i+1} \wedge \cdots \wedge a^k.$$  

Each $k$-covector in the sum on the right-hand side of this equation must be zero, since $j$ must appear twice among the upper indices. This shows that $a^1 \wedge \cdots \wedge a^k = 0$ whenever the 1-covectors are linearly dependent.

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Now let us suppose that the \( a^i \) are linearly independent. By choose additional 1-covectors we may form a basis of the dual space \( V^* \): \( \{ a^1, \ldots, a^k, a^{k+1}, \ldots, a^n \} \) where, naturally, \( n \) is the dimension of \( V^* \), which is also the dimension of \( V \). This basis of \( V^* \) defines uniquely a dual basis of \( V \): \( u_1, \ldots, u_n \), so that \( a^i(u_j) = \delta_{ij} \). Then

\[
(a^1 \wedge \cdots \wedge a^n)(u_1, \ldots, u_n) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a^1(u_{\sigma(1)}) \cdots a^n(u_{\sigma(n)}) = a^1(u_1) \cdots a^n(u_n) = 1.
\]

We conclude that \( a^1 \wedge \cdots \wedge a^n \) must not be zero.

\[\diamond \]

6. **Problem 3.11, page 33, Tu. Exterior multiplication.** Let \( \alpha \) be a nonzero 1-covector and \( \gamma \) a \( k \)-covector on a finite-dimensional vector space \( V \). Show that \( \alpha \wedge \gamma = 0 \) if and only if \( \gamma = \alpha \wedge \beta \) for some \((k-1)\)-covector \( \beta \) on \( V \).

**Solution.** If \( \gamma = \alpha \wedge \beta \) then \( \alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta \), which is 0 since \( \alpha \wedge \alpha \) is 0 for all 1-covector \( \alpha \). Now suppose that \( \alpha \wedge \gamma = 0 \). Let \( \{ a^1, \ldots, a^n \} \) be a basis of \( L_1(V) \) such that \( a^1 = \alpha \). We know that the set of all \( a^{i_1} \wedge \cdots \wedge a^{i_k} \), \( i_1 < \cdots < i_k \), forms a basis of space of \( k \)-covectors. Let us partition this basis into two disjoint subsets: \( A_1 \), consisting of those basis elements for which \( i_1 = 1 \); and \( A_2 \), consisting of those basis elements for which \( i_1 > 1 \). Let us write \( \gamma = \gamma_1 + \gamma_2 \), where \( \gamma_i \) is a linear combination of elements in \( A_i \). Thus \( \gamma_1 = \alpha \wedge \beta \), for some \((k-1)\)-covector \( \beta \) and

\[
\sum_{1 < i_1 < \cdots < i_k} a_{i_1 \ldots i_k} a^{i_1} \wedge \cdots \wedge a^{i_k}.
\]

Now

\[
0 = \alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta + \alpha \left( \sum_{1 < i_1 < \cdots < i_k} a_{i_1 \ldots i_k} a^{i_1} \wedge \cdots \wedge a^{i_k} \right) = \sum_{1 < i_1 < \cdots < i_k} a_{i_1 \ldots i_k} \alpha \wedge a^{i_1} \wedge \cdots \wedge a^{i_k}.
\]

But the sum in the far right term of these identities involves linearly independent covectors, so all the coefficients \( a_{i_1 \ldots i_k} \) are zero. We conclude that \( \gamma = \alpha \wedge \beta \).

\[\diamond \]

7. **Tu’s book, page 34-40.** Read Chapter 1, Sections 4.1, 4.2, and 4.3.

\[\diamond \]

8. **Problem 4.2, page 44, Tu. A 2-form on \( \mathbb{R}^3 \).** At each point \( p \in \mathbb{R}^3 \), define a bilinear function \( \omega_p \) on \( T_p \mathbb{R}^3 \) by

\[
\omega_p(a, b) = \omega_p \left( \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} \right) = p^3 \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \\ a^3 & b^3 \end{bmatrix}.
\]

for tangent vectors \( a, b \in T_p \mathbb{R}^3 \), where \( p^3 \) is the third component of \( p = (p^1, p^2, p^3) \). Since \( \omega_p \) is an alternating bilinear function on \( T_p \mathbb{R}^3 \), \( \omega \) is a 2-form on \( \mathbb{R}^3 \). Write \( \omega \) in terms of the standard basis \( dx^1 \wedge dx^j \) at each point.
Solution. Let $\Omega$ be the 2-form given by

$$\Omega = x^3 \, dx^1 \wedge dx^2.$$ 

Then a simple evaluation shows that $\Omega_p(a, b) = \omega_p(a, b)$.

9. Problem 4.4, page 45, Tu. Spherical coordinates. Suppose the standard coordinates on $\mathbb{R}^3$ are called $\rho, \phi$ and $\theta$. If 

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

calculate $dx, dy, dz$ and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi,$ and $d\theta$.

Solution. We have 

$$dx = \sin \phi \cos \theta \, d\rho + \rho \cos \phi \cos \theta \, d\phi - \rho \sin \phi \sin \theta \, d\theta,$$
$$dy = \sin \phi \sin \theta \, d\rho + \rho \cos \phi \sin \theta \, d\phi + \rho \sin \phi \cos \theta \, d\theta,$$
$$dz = \cos \phi \, d\rho - \rho \sin \phi \, d\phi.$$ 

Using the basic algebraic properties of $\wedge$ (namely, $\alpha \wedge \alpha = 0$ and $\alpha \wedge \beta = -\beta \wedge \alpha$ for 1-covectors $\alpha$ and $\beta$, in addition to associativity and distributivity of $\wedge$) and the trigonometric identity $\sin^2 \phi + \cos^2 \phi = 1$, 

$$(dx \wedge dy) \wedge dz = (\rho \sin^2 \phi \, d\rho \wedge d\theta + \rho^2 \sin \phi \cos \phi \, d\phi \wedge d\theta) \wedge (\cos \phi \, d\rho - \rho \sin \phi \, d\phi) = \rho^2 \sin \phi \, d\rho \wedge d\phi \wedge d\theta.$$

10. Problem 4.5, page 45, Tu. Wedge product. Let $\alpha$ be a 1-form and $\beta$ a 2-form on $\mathbb{R}^3$. Then 

$$\alpha = a_1 \, dx^1 + a_2 \, dx^2 + a_3 \, dx^3,$$
$$\beta = b_1 \, dx^2 \wedge dx^3 + b_2 \, dx^3 \wedge dx^1 + b_3 \, dx^1 \wedge dx^2.$$ 

Simplify the expression $\alpha \wedge \beta$ as much as possible.

Solution. By straightforward algebra, using the properties of $\wedge$, we find 

$$\alpha \wedge \beta = (a_1 b_1 + a_2 b_2 + a_3 b_3) \, dx^1 \wedge dx^2 \wedge dx^3.$$

\[ \diamond \]