1. **Homogeneous and decomposable covectors.** Let $V$ be a finite-dimensional vector space. An element $\tau \in A_\ast(V)$ (the exterior algebra on $V$; see page 30 of Tu) is said to be **homogeneous** of degree $k$ if $\tau \in A_k(V)$, and a homogeneous element of degree $k \geq 1$ is said to be **decomposable** if there exist $\alpha_1, \ldots, \alpha_k \in A_1(V)$ such that $\tau = \alpha_1 \wedge \cdots \wedge \alpha_k$.

(a) If $\tau \in A_k(V)$ is decomposable, what is $\tau \wedge \tau$? (Explain.)

(b) If $\dim V > 3$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are linearly independent, is $\alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4$ decomposable?

(c) Prove that if $\dim V = n \leq 3$, then every homogenous element of degree $k \geq 1$ is decomposable.

(d) If $\dim V = 4$, give an example of a non-decomposable homogeneous of $A_\ast(V)$.

2. **A useful identity for the exterior derivative of 1-forms.** Let $\omega$ be a differential 1-form on an open subset $U \subset \mathbb{R}^n$. and let $X, Y$ be smooth vector fields on $U$. Show that

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

3. **Closed and exact forms.** Determine which of the following differential forms on $\mathbb{R}^3$ are closed and which are exact. If the form is exact, write it as the exterior derivative of another form. (Note: At this stage in the theory, solving the system of partial differential equations $\gamma = d\beta$ for the coefficient functions of $\beta$, when $\gamma$ is closed, is guesswork. We will see later, when we get to Poincaré’s lemma, that this equation is always solvable locally, and we will have a general method for finding the solution.)

(a) $\alpha = xy\,dx + xz\,dy + xy\,dz$

(b) $\beta = x\,dx + x^2\,y^2\,dy + yz\,dz$

(c) $\gamma = 2xy^2\,dx \wedge dy + yz\,dy \wedge dz$. (Hint: Try $\gamma = d\theta$ for $\theta = h\,dy$.)

4. **A property of the wedge product.** (You may find this problem useful for solving Problem 5. It is not necessary to turn it in with the rest of the homework but think about it and, if you don't figure it out, make sure to read my solution later.) Let $\alpha = a^1 \wedge \cdots \wedge a^k$ be a decomposable element of $A_k(V)$ and $u_1, \ldots, u_k \in V$. Show that

$$\left(a^1 \wedge \cdots \wedge a^k\right)(u_1, \ldots, u_k) = \sum_i (-1)^{i-1} a^i(u_1)\left(a^1 \wedge \cdots \wedge a^i \wedge a^{i+1} \wedge \cdots \wedge a^k\right)(u_2, \ldots, u_k).$$

5. **An antiderivation of degree $-1$.** Recall the notion of antiderivation of degree $m$ (Definition 4.6, page 39, of Tu’s text). Consider the graded algebra of covectors $A_\ast(V)$ on a finite dimensional vector space $V$. (Page 30 of Tu’s text.) Given any $v \in V$, define the linear map $i_v : A_\ast(V) \rightarrow A_\ast(V)$, known as the **interior product**, so that $i_v$ maps $A_0(V)$ to 0, and for all $\omega \in A_k(V)$, $k \geq 1$,

$$(i_v \omega)(u_1, \ldots, u_{k-1}) = \omega(v, u_1, \ldots, u_{k-1}).$$
Show that the interior multiplication $i_x$ is an antiderivation of degree $-1$.

6. **The Lie derivative on forms.** The Lie derivative is a very useful operation on tensor fields of general type. Here we define it on differential forms. A more conceptual characterization will be provided later. Given a smooth vector field $X \in \mathfrak{X}(U)$ on an open set $U \subset \mathbb{R}^n$, we define $\mathcal{L}_X : \Omega^*(U) \to \Omega^*(U)$ by the following properties:

- $\mathcal{L}_X f = X f$ for all $f \in C^\infty(U)$;
- $\mathcal{L}_X (\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \left( \mathcal{L}_X \beta \right)$ for any differential forms $\alpha, \beta$;
- $\mathcal{L}_X d = d \mathcal{L}_X$.

Thus $\mathcal{L}_X$ is a derivation of degree 0 on differential forms that commutes with the exterior derivative and agrees with the directional derivative on smooth functions. It is not difficult to check that these three conditions uniquely and consistently determine $\mathcal{L}_X$. (It is not necessary to prove this here.) Use these properties to find the Lie derivative of $\omega = dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ along the vector field $X = f_1 \frac{\partial}{\partial x^1} + \cdots + f_n \frac{\partial}{\partial x^n}$. (Note: since the Lie derivative preserves the degree of the form, we must have $\mathcal{L}_X \omega = q \omega$ for some smooth function $q$.) What is this function in terms of $X$? It corresponds to a well-known operation on vectors in vector calculus.

7. **Pull-back of differential forms.** *Only do parts (e) and (f).* We will return to this concept later in class. (This material is in Chapter 5, Section 18.5, page 204 of Tu, where the concept of pull-back of differential forms is introduced directly on smooth manifolds rather than first on $\mathbb{R}^n$ as we do here.) Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map, where $U$ is an open subset. Let $\alpha$ be a differential $k$-form defined on a subset of $\mathbb{R}^n$ that contains the image of $f$. We define the pull-back of $\alpha$ under $f$ to be the $k$-form $f^* \alpha$ on $U$ given at each $p \in U$ by

$$ (f^* \alpha)_p(v_1, \ldots, v_k) = \alpha_{f(p)}(df_p v_1, \ldots, df_p v_k). $$

Recall that the differential at $p$ of a map $f$ is the linear map $df_p : T_p \mathbb{R}^m \to T_{f(p)} \mathbb{R}^n$ defined by $df_p v = \frac{df}{dt}|_{t=0} f(\gamma(t))$ where $\gamma(t)$ is any differentiable path such that $\gamma(0) = p$ and $\gamma'(0) = v$. (As a matter of notation, notice that, when $n = 1$, there are two ways of interpreting $df_p$: one as the linear map $df_p : T_p \mathbb{R}^m \to T_{f(p)} \mathbb{R}^1$ and the other as the differential 1-form obtained by applying the exterior derivative $d$ to $f$. These two interpretations agree under the natural isomorphism $T_{f(p)} \mathbb{R}^1 \cong \mathbb{R}$.) When $\alpha$ is a differential 0-form, namely, a smooth function, we define the pull-back as $f^* \alpha = \alpha \circ f$. We then extend this definition from general $k$-forms to mixed degree forms in $\Omega^*(U)$.

(a) Show that $f^*$ is a linear map on differential $k$-forms.
(b) Show that $f^*$ respects the wedge product: $f^* (\alpha \wedge \beta) = f^* \alpha \wedge f^* \beta$.
(c) Show that $f^*$ commutes with the exterior derivative: $df^* \alpha = f^* d\alpha$.
(d) Show that $f^* \left( \sum_i h_i dx^i \right) = \sum_i h_i \circ f du^i$ where $u^i = x^i \circ f$.
(e) Find $f^* \alpha$ where $\alpha = (2xy + x^2 + 1) \, dx + (x^2 - y) \, dy$ and $f : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$ (u, v, w) \mapsto (x, y) = \left( u - v, v^2 + w \right). $$

(f) Find $f^* \omega$ where $\omega = dx \wedge dy \wedge dz$ and $f : (0, \infty) \times (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$ is given by

$$ f : (\rho, \theta, \varphi) \mapsto (x, y, z) = \left( \rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi \right). $$