1. Read Sections §6 and §7, pages 59 – 81, of Tu.

2. Problem 6.1, page 70, Tu. Differentiable structures on $\mathbb{R}$. Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \phi = \text{Id} : \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \psi : \mathbb{R} \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (Hint: The identity map $\mathbb{R} \to \mathbb{R}'$ is not the desired diffeomorphism; in fact, this map is not smooth.)

Solution.

(a) To show that the two structures are distinct, it suffices to show that the two charts are not $C^\infty$ compatible. That is, it suffices to show that $\psi \circ \phi^{-1}$ is not smooth. Now, $(\psi \circ \phi^{-1})(x) = x^{1/3}$, which is not differentiable at 0. Thus the two charts define different smooth structures.

(b) Nevertheless, the two smooth structures are diffeomorphic. In fact, let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = x^3$. Then $\psi \circ F \circ \phi^{-1} : x \mapsto (x^3)^{1/3} = x$. Similarly, $\phi \circ F^{-1} \circ \psi^{-1} : x \mapsto (x^3)^{1/3} = x$. These are the identity map on $\mathbb{R}$, which is $C^\infty$. Therefore $F$ is a diffeomorphism.

3. Problem 6.2, page 70, Tu. The smoothness of the inclusion map. Let $M$ and $N$ be manifolds and let $q_0$ be a point in $N$. Prove that the inclusion map $i_{q_0} : M \to M \times N$, $i_{q_0}(p) = (p, q_0)$, is $C^\infty$.

Solution. It suffices to prove that the inclusion map is $C^\infty$ about $p$ for an arbitrary $p \in M$. Let $(U, \phi)$ be a smooth chart for $M$ such that $p \in U$, and let $(V, \psi)$ be a smooth chart for $N$ such that $q_0 \in V$. Then $(U \times V, \phi \times \psi)$ is a smooth chart for the product manifold $M \times N$ such that $(p, q_0) \in U \times V$. Without loss of generality we may assume that $\phi(p) = 0 \in \mathbb{R}^m$ and $\psi(q_0) = 0 \in \mathbb{R}^n$. Now note: $F = (\phi \times \psi) \circ i_{q_0} \circ \phi^{-1} : \phi(U) \to \phi(U) \times \psi(V)$ satisfies $F(x) = (x, 0)$, which is $C^\infty$ at $x = 0$. 

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4. **Problem 7.4, page 81, Tu. Quotient of a sphere with antipodal points identified.** Let $S^n$ be the unit sphere centered at the origin in $\mathbb{R}^{n+1}$. Define an equivalence relation $\sim$ on $S^n$ by identifying antipodal points:

$$x \sim y \iff x = \pm y, \ x, y \in S^n.$$

(a) Show that $\sim$ is an open equivalence relation.

(b) Apply Theorem 7.7 and Corollary 7.8 to prove that the quotient space $S^n / \sim$ is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \cong S^n / \sim$.

**Solution.**

(a) Let $\pi : S^n \to S^n / \sim$ be the projection map. If $U \subset S^n$ is an open set, $\pi^{-1} \pi(U)$ is the union of $U$ and $-U := \{-x : x \in S^n\}$. Being the union of two open sets, $\pi^{-1} \pi(U)$ is open. Therefore $\sim$ is an open equivalence relation.

(b) The graph of the equivalence relation $\sim$ is the union $R = \Delta^+ \cup \Delta^-$ where

$$\Delta^\pm = \{(x, y) \in S^n \times S^n : y = \pm x\}.$$ 

Since $\Delta^\pm$ is the graph of the continuous map $x \in S^n \mapsto \pm x \in S^n$, it follows that $\Delta^\pm$ is closed. By Theorem 7.7 the quotient $S^n / \sim$ is Hausdorff.

5. **Problem 7.4, page 81, Tu. Orbit space of a continuous group action.** Suppose a right action of a topological group $G$ on a topological space $S$ is continuous; this simply means that the map $S \times G \to S$ describing the action is continuous. Define two points $x, y \in S$ to be equivalent if they are in the same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let $S/G$ be the quotient space; it is called the *orbit space* of the action. Prove that the projection map $\pi : S \to S / G$ is an open map. (This problem generalizes Proposition 7.14, in which $G = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. Because $\mathbb{R}^\times$ is commutative, a left $\mathbb{R}^\times$-action becomes a right $\mathbb{R}^\times$-action if scalar multiplication is written on the right.)

**Solution.** To show the projection map is open, it suffices to show that, for any open set $U \subset S$, the *saturation* of $U$, defined as $\pi^{-1} \pi(U)$, is also open in $S$. First note that

$$\pi^{-1} \pi(U) = \{x \in S : xg^{-1} \in U \text{ for some } g \in G\} = \pi^{-1} \pi(U) = \bigcup_{g \in G} Ug.$$

But $Ug$ is open for each $g \in G$ since the action is continuous and $Ug$ is the inverse image of $U$ under the continuous map $x \in S \mapsto xg^{-1} \in S$. Therefore $\pi^{-1} \pi(U)$ is the union of open sets, hence open.

6. **Problem 7.8, page 82, Tu. The Grassmannian $G(k, n)$.** *(You do not need to submit answers to this problem, but work it out and familiarize yourself with the Grassmannian since I may use it as an example of manifold later on.)*

The Grassmannian $G(k, n)$ is the set of all $k$-planes through the origin in $\mathbb{R}^n$. Such a $k$-plane is a linear subspace
of dimension \( k \) of \( \mathbb{R}^n \) and has a basis consisting of \( k \) linearly independent vectors \( a_1, \ldots, a_k \) in \( \mathbb{R}^n \). It is therefore completely specified by an \( n \times k \) matrix \( A = [a_1 \cdots a_k] \) of rank \( k \), where the rank of a matrix \( A \), denoted by \( \text{rank} A \), is defined to be the number of linearly independent columns of \( A \). This matrix is called a matrix representative of the \( k \)-plane. (For properties of the rank, see the problems in Appendix B.)

Two bases \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \) determine the same \( k \)-plane if there is a change-of-coordinate matrix \( g = [g_{ij}] \in GL(k, \mathbb{R}) \) such that

\[
b_j = \sum_i a_i g_{ij}, \quad 1 \leq i, j \leq k.
\]

In matrix notation, \( B = Ag \).

Let \( F(k, n) \) be the set of all \( n \times k \) matrices of rank \( k \), topologized as a subspace of \( \mathbb{R}^{n \times k} \), and \( \sim \) the equivalence relation

\[
A \sim B \quad \text{if and only if} \quad \text{there is a matrix } g \in GL(k, \mathbb{R}) \text{ such that } B = Ag.
\]

In the notation of Problem B.3, \( F(k, n) \) is the set \( D_{\text{max}} \) in \( \mathbb{R}^{n \times k} \) and is therefore an open subset. There is a bijection between \( G(k, n) \) and the quotient space \( F(k, n) / \sim \). We give the Grassmannian \( G(k, n) \) the quotient topology on \( F(k, n) / \sim \).

(a) Show that \( \sim \) is an open equivalence relation. (Hint: Either mimic the proof of Proposition 7.14 or apply Problem 7.5.)

(b) Prove that the Grassmannian \( G(k, n) \) is second countable. (Hint: Apply Corollary 7.10.)

(c) Let \( S = F(k, n) \). Prove that the graph \( R \) in \( S \times S \) of the equivalence relation \( \sim \) is closed. (Hint: Two matrices \( A = [a_1 \cdots a_k] \) and \( B = [b_1 \cdots b_k] \) in \( F(k, n) \) are equivalent if and only if every column of \( B \) is a linear combination of the columns of \( A \) if and only if \( \text{rank}(AB) \leq k \) if and only if all \( (k + 1) \times (k + 1) \) minors of \( [AB] \) are zero.)

(d) Prove that the Grassmannian \( G(k, n) \) is Hausdorff. (Hint: Mimic the proof of Proposition 7.16.)

Next we want to find a \( C^\infty \) atlas on the Grassmannian \( G(k, n) \). For simplicity, we specialize to \( G(2, 4) \). For any \( 4 \times 2 \) matrix \( A \) let \( A_{ij} \) be the \( 2 \times 2 \) submatrix consisting of its \( i \)th row and \( j \)th row. Define

\[
V_{ij} = \{ A \in F(2, 4) : A_{ij} \text{ is nonsingular} \}.
\]

Because the complement of \( V_{ij} \) in \( F(2, 4) \) is defined by the banishing of \( \det A_{ij} \), we conclude that \( V_{ij} \) is an open subset of \( F(2, 4) \).

(e) Prove that if \( A \in V_{ij} \), then \( Ag \in V_{ij} \) for any nonsingular matrix \( g \in GL(2, \mathbb{R}) \). Define \( U_{ij} = V_{ij} / \sim \). Since \( \sim \) is an open equivalence relation, \( U_{ij} \) is an open subset of \( G(2, 4) \). For \( A \in V_{1,2} \),

\[
A \sim AA_{12}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{pmatrix} = \begin{pmatrix} I \\ A_4 A_{12}^{-1} \end{pmatrix}.
\]

This shows that the matrix representatives of a \( 2 \)-plane in \( U_{12} \) have a canonical form \( B \) in which \( B_{12} \) is the identity matrix.

(f) Show that the map \( \tilde{\phi}_{12} : V_{12} \to \mathbb{R}^{2 \times 2} \),

\[
\tilde{\phi}_{12}(A) = A_{34} A_{12}^{-1}
\]

induces a homeomorphism \( \phi_{12} : U_{12} \to \mathbb{R}^{2 \times 2} \).
(g) Define similarly homeomorphisms \( \phi_{ij} : U_{ij} \to \mathbb{R}^{2 \times 2} \). Compute \( \phi_{12} \circ \phi_{23}^{-1} \) and show that it is \( C^\infty \).

(h) Show that \( \{U_{ij} : 1 \leq i < j \leq 4 \} \) is an open cover of \( G(2,4) \) and that \( G(2,4) \) is a smooth manifold.

Similar consideration shows that \( F(k,n) \) has an open cover \( \{V_I\} \), where \( I \) is a strictly ascending multi-index \( 1 \leq i_1 < \cdots < i_k \leq n \). For \( A \in F(k,n) \), let \( A_I \) be the \( k \times k \) submatrix of \( A \) consisting of \( i_1 \)th, \( i_2 \)th, \( i_3 \)th rows of \( A \). Define
\[
V_I = \{ A \in G(k,n) : \det A_I \neq 0 \}.
\]

Next define \( \tilde{\phi}_I : V_I \to \mathfrak{gl}(n-k) \times k \) by
\[
\tilde{\phi}_I(A) = (AA_I^{-1})_{I'}
\]
where \( (\cdot)_{I'} \) denotes the \( (n-k) \times k \) submatrix obtained from the complement \( I' \) of the multi-index \( I \). Let \( U_I = V_I/\sim \). Then \( \tilde{\phi} \) induces a homeomorphism \( \phi : U_I \to \mathbb{R}^{(n-k) \times k} \). It is not difficult to show that \( \{(U_I, \phi_I)\} \) is a \( C^\infty \) atlas for \( G(k,n) \). Therefore the Grassmannian \( G(k,n) \) is a \( C^\infty \) manifold of dimension \( k(n-k) \).

**Solution.**

(a) An equivalence relation \( \sim \) on a topological space \( S \) is said to be **open** if the quotient map \( \pi : S \to S/\sim \) is an open map. Equivalently, \( \pi^{-1}(\pi(V)) \) is an open set for all open \( V \subset S \). Let \( V \subset F(k,n) \) be an open set and consider \( \tilde{V} := \pi^{-1}(\pi(V)) \), where \( \pi : F(k,n) \to G(k,n) \) is the quotient map. Then a matrix \( B \) belongs to \( \tilde{V} \) if and only if \( B = Ag \) for some \( A \in V \) and \( g \in GL(k,\mathbb{R}) \). This means that
\[
\tilde{V} = \bigcup_{g \in GL(k,\mathbb{R})} Vg.
\]
Each \( Vg \) is open in \( F(k,n) \) since left-multiplication by \( g \) is a homeomorphism. Being the union of open sets, \( \tilde{V} \) is open in \( F(k,n) \). Therefore \( \sim \) is an open equivalence relation.

(b) \( F(k,n) \) is second countable since it may be regarded as a subset of \( \mathbb{R}^{k \times n} \), which is second countable. By Corollary 7.10, page 76 of the textbook, \( G(k,n) \) is second countable since it is the quotient space of the second countable space \( F(k,n) \) under the open equivalence relation \( \sim \).

(c) If \( A, B \in F(k,n) \), then
\[
A \sim B \iff B = Ag, \text{ some } g \in GL(k,\mathbb{R})
\]
\[
\iff [AB] (a n \times 2k \text{-matrix}) \text{ has rank } k
\]
\[
\iff [AB] \text{ has rank } \leq k
\]
\[
\iff \text{ all } (k+1) \times (k+1) \text{ minors } = 0.
\]

Thus
\[
R = \{(A,B) \in S \times S : \text{ all } (k+1) \times (k+1) \text{ minors of } [AB] \text{ are } 0\}.
\]

Therefore, as the minors are continuous functions of the entries of the matrices, \( R \) is a closed subset of \( S \times S \) and the relation is closed.

(d) This is now immediate from what has been proved above and Theorem 7.7, which says that if the relation \( \sim \) is open and \( R \subset S \times S \) is closed then \( S/\sim \) is Hausdorff.

(e) Instead of restricting to the \( (k = 2, n = 4) \) case, let us prove the rest of the exercise in general, following the lines indicated on page 83 of Tu’s text. Let \( I = (i_1, \ldots, i_k), \ 1 \leq i_1 < \cdots < i_k \leq n \), be a strictly ascending multi-index. For \( A \in F(k,n) \), let \( A_I \) be the \( k \times k \) submatrix of \( A \) consisting of the \( i_1 \)th, \( i_2 \)th, \( i_3 \)th rows of \( A \).
Define
\[ V_I = \{ A \in F(k, n) : \det A_I \neq 0 \}. \]

Now, if \( A \in V_I \) and \( g \in GL(k, \mathbb{R}) \), then
\[ \det(Ag)_I = \det(A_I g) = \det A_I \det g \neq 0. \]

Therefore \( Ag \in V_I \).

Next define \( \tilde{\phi}_I : V_I \to \mathbb{R}^{(n-k)\times k} \) by \( \tilde{\phi}_I(A) = \left( A A_I^{-1} \right)_I = A_I g A_I^{-1} \), where \((\ )_I \) denotes the \((n-k) \times k \) submatrix obtained from the complement multi-index \( I' \) of \( I \). Let \( U_I = V_I/\sim \). We wish to show that \( \tilde{\phi}_I \) induces a homeomorphism \( \phi_I : U_I \to \mathbb{R}^{(n-k)\times k} \).

Let \( A, B \in V_I \) be equivalent, so \( B = Ag \) for some \( g \in GL(k, \mathbb{R}) \). This implies \( B_I = A_I g \) hence \( g = A_I^{-1} B_I \). Therefore
\[ \tilde{\phi}_I(B) = BB_I^{-1} = AA_I^{-1} = \tilde{\phi}_I(A). \]

This shows that there exists \( \phi_I : U_I \to \mathbb{R}^{(n-k)\times k} \) such that \( \tilde{\phi}_I = \phi_I \circ \pi \), where \( \pi : V_I \to U_I \) is the quotient (projection) map. As \( \tilde{\phi}_I \) is continuous, \( \phi_I \) is also continuous, and as \( V_I \) is open and the relation is open, \( U_I \) is open in the quotient topology.

To see that \( \phi_I \) is a bijection, note that if \( A, B \in F(k, n) \) are elements of \( V_I \) such that \( \tilde{\phi}_I(A) = \tilde{\phi}_I(B) \), then \( AA_I^{-1} = BB_I^{-1} \); but then \( B = Ag \) where \( g = A_I^{-1} B_I \in GL(k, \mathbb{R}) \). This means that \( A \sim B \), hence define the same element in \( U_I \).

The inverse map of \( \phi_I \) can be given explicitly: For each \( C \in \mathbb{R}^{(n-k)\times k} \) (viewed as a matrix with \( n-k \) rows and \( k \) columns), let \( \tilde{C} \) be the \( n \times k \) matrix such that \( \tilde{C}_I = C \) and \( \tilde{C} \) is the \( k \times k \) identity matrix. Then \( C \sim \pi(\tilde{C}) \) is continuous and it is the inverse map of \( \phi \) since \( \pi(\tilde{C}) \) is mapped to \( C \) under \( \phi_I \).

Let \( J = (j_1, \ldots, j_k) \), \( 1 \leq j_1 < \cdots < j_k \leq n \), be another multi-index. Let \( X \in \phi_I(U_I \cap U_J) \). Define \( \tilde{X}^{(I)} \) as the \( n \times k \) matrix such that \( \tilde{X}^{(I)}_I = X \) and \( \tilde{X}^{(I)} \) is the \( k \times k \) identity, and let \( X^{(I)} = \pi(\tilde{X}^{(I)}) \). By definition, \( \tilde{X}^{(I)} \in V_I \) and \( X^{(I)} \in U_I \). Then \( X^{(I)} \) is the unique element of \( U_I \) mapping to \( X \) under the homeomorphism \( \phi_I \). Since \( X \in \phi_I(U_I \cap U_J) \), we must have \( X^{(I)} \in U_I \cap U_J \). This means that \( \tilde{X}^{(I)} \) is invertible and
\[ Y := \phi_I(\tilde{X}^{(I)^{-1}}(X)) = \tilde{X}^{(I)} \left( \tilde{X}^{(I)}_I \right)^{-1}. \]

This shows that \( Y \) is a smooth function of \( X \), hence \( \phi_I \circ \phi_I^{-1} : \phi_I(U_I \cap U_J) \to \phi_I(U_I \cap U_J) \) is a \( C^\infty \) diffeomorphism.

The collection of sets \( U_I \) covers \( G(k, n) \) since every element of \( F(k, n) \) has a no-zero \( k \times k \) minor. Therefore, the collection of all \((U_I, \phi_I)\) defines a \( C^\infty \) atlas, making \( G(k, n) \) a smooth manifold.

\( \diamond \)