Local properties of smooth maps

Let us summarize here the main results pertaining to the local properties of smooth maps between Euclidean spaces. All functions and maps will be assumed here to be smooth (infinitely continuously differentiable). By a *neighborhood* of a point I mean an open set containing the point. A map \( f : U \rightarrow V \) is said to be a smooth *diffeomorphism* between open subsets \( U, V \) of \( \mathbb{R}^n \) if it is bijective and both \( f \) and \( f^{-1} \) are smooth maps; a map \( f : U \rightarrow \mathbb{R}^n \) where \( U \) is an open subset of \( \mathbb{R}^n \) is said to be a *local diffeomorphism* if for each \( p \in U \) there are open neighborhoods \( V \subset U \) and \( W \) of \( f(p) \) such that the restriction \( f|_V : V \rightarrow W \) is a diffeomorphism.

**Theorem 1** (Inverse function theorem). Let \( f : U \rightarrow \mathbb{R}^n \) be a smooth map where \( U \) is an open and non-empty subset of \( \mathbb{R}^n \). Suppose that at some \( p \in U \) the differential \( d f_p \) is a linear isomorphism. Then \( f \) is a smooth diffeomorphism from a neighborhood \( V \) of \( p \) onto a neighborhood \( W \) of \( f(p) \).

**Corollary 2.** Let \( U \subset \mathbb{R}^n \) be open and \( f : U \rightarrow \mathbb{R}^n \) be a smooth map. Then \( f \) is a local diffeomorphism if and only if for each \( p \in U \) the differential \( d f_p \) is an isomorphism.

![Figure 1](image)

Figure 1: By the implicit function theorem, a change of coordinates \( h \) in turns a submersion \( f \) locally into a projection \( \pi \).

A differentiable map \( f : U \rightarrow \mathbb{R}^m \) from the open subset \( U \) of \( \mathbb{R}^n \) is said to be a *submersion* if for each \( p \in U \) the diffeomorphism \( d f_p \) is surjective. This is only possible when \( n \geq m \). It is not difficult to show that if a smooth map \( f \) has the property that \( d f_p \) is surjective for some \( p \), then \( d f_q \) is surjective for all \( q \) in some sufficiently small neighborhood of \( p \). The next theorem generalizes the inverse function theorem.
Theorem 3 (Local form of submersions). Let $U \subset \mathbb{R}^{m+n}$ be an open set and $f : U \to \mathbb{R}^n$ a smooth map. Suppose that at $p \in U$ the differential $df_p$ is surjective. Then, after possibly replacing $U$ with a smaller neighborhood of $p$, we can find a neighborhood $V$ of $0$ in $\mathbb{R}^m$, a neighborhood $W$ of $f(p)$ in $\mathbb{R}^n$, and a diffeomorphism $h : V \times W \to U$ such that $f \circ h = \pi$, where $\pi : V \times W \to W$ is the projection $\pi(q,w) = w$.

This means that if $df_p$ is surjective at some $p$, we can find a coordinate change (a diffeomorphism) of a neighborhood of $p$ so that in the new coordinates $f$ becomes the projection $\pi(p_1, p_2) = p_2$. A consequence of this fact is that the set of points in this neighborhood of $p$ that are mapped under $f$ to $q = f(p)$ (that is, the level set of $f$ at the value $q$) is diffeomorphic to a neighborhood of $0$ in $\mathbb{R}^m$. Said in yet a different way, when $df_p$ is surjective, the level set of $f$ for the value $q = f(p)$ admits near $p$ a smooth parametrization. The main consequence for us is this:

Theorem 4. Let $U$ be an open subset of $\mathbb{R}^{n+m}$ and $f : U \to \mathbb{R}^n$ a smooth map. Let $q \in \mathbb{R}^n$ and suppose that $df_p$ is surjective for all $p \in f^{-1}(q) \cap U$. Then the level set $M = \{ p \in U : f(p) = q \}$ is a smooth submanifold of $U$.

A smooth map $f$ from an open set $U \subset \mathbb{R}^m$ into $\mathbb{R}^n$ is said to be an immersion if for every $p \in U$ the differential $df_p$ is injective. Note that if $df_p$ is injective, then $df_q$ is injective for all $q$ in some neighborhood of $p$.

Theorem 5 (Local form of immersions). Let $f$ be a smooth map from an open subset of $\mathbb{R}^m$ into $\mathbb{R}^{n+m}$ such that at a point $p$ the differential $df_p$ is injective. Then there exist neighborhoods $V$ of $p$, $U$ of $f(p)$, $W$ of $0$ in $\mathbb{R}^n$, and a diffeomorphism $h : U \to V \times W$ such that $h \circ f = i$ where $i(p) = (p,0)$.

Figure 2: After a change of coordinates $h$ the immersion $f$ becomes the inclusion map $i : p \mapsto (p,0)$. In particular, $h^{-1} \circ i$ is a smooth parametrization of the image of $V$ under $f$ on some neighborhood of $f(p)$.

The above theorems can be viewed as special cases of the following.

Theorem 6 (Constant rank theorem). Let $U$ be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ a smooth map having constant rank equal to $k$ in a neighborhood of $p \in U$. Then there is a diffeomorphism $h$ of a neighborhood of $p$ in $\mathbb{R}^n$ and a diffeomorphism $g$ of a neighborhood of $f(p)$ in $\mathbb{R}^m$ sending $p$ to the origin on $\mathbb{R}^n$ and $f(p)$ to the origin of $\mathbb{R}^m$ such that

$$(g \circ f \circ h^{-1})(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

Being local in nature, the above theorems naturally extend to similar statements on smooth manifolds. A especially useful form of the Implicit Function Theorem is the following. (See Tu, Theorem 9.9, page 105.)
Theorem 7 (Implicit function theorem on manifolds). Let $F : N \to M$ be a smooth map of manifolds, with $\dim N = n$ and $\dim M = m$. Let $q \in M$ be a regular value of $F$ whose level set is nonempty. Then the level set $F^{-1}(\{q\})$ is a regular submanifold of $N$ of dimension equal to $n - m$.

Lie groups

A manifold $G$ which is also a group so that the group multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are smooth maps is said to be a Lie group. For any given $g \in G$, let $L_g : G \to G$ denote the left translation map, defined by $L_g(h) = gh$. Then it is easy to verify that $L_g$ is a diffeomorphism from $G$ to itself. The same is true for the right translation map $R_g(h) = hg$. A vector field $X$ on $G$ is said to be left-invariant if for all $g, h \in G$

$$(dL_g)_h X(h) = X(gh).$$

Here $X(h) \in T_h G$ indicates the value of the vector field at $h$. A left-invariant vector field is completely determined by its value at $e$. Thus the collection of all left-invariant vector fields on $G$ is a vector space of the same dimension as $T_e G$, called the Lie algebra of $G$. We denote the Lie algebra of $G$ by $\mathfrak{g}$.

Quotient manifolds

Suppose that $G$ is a Lie group, possibly 0-dimensional. Let $M$ be a smooth manifold. A smooth map

$$\rho : G \times M \to M$$

is said to be a smooth action of $G$ on $M$ if $g \mapsto \rho(g, \cdot)$ is a group homomorphism of $G$ into the group of all diffeomorphisms of $M$. Thus, if we write $\rho_g : M \to M$ so that $\rho_g(p) = \rho(g, p)$, then $\rho_e = \text{id}_M$ and $\rho_{g_1 g_2} = \rho_{g_1} \circ \rho_{g_2}$. If follows from these that $\rho_{g}^{-1} = \rho_{g^{-1}}$. The group action is said to be free if $\rho_g(p) = p$ for some $p$ implies that $g = e$. In other words, no element of $G$ other than the identity fixes any point of $M$. We often use the simpler notation $g p = \rho_g(p)$. We say that the action is proper if the map

$$(g, p) \in G \times M \to (g p, p) \in M \times M$$

is a proper map, which means that the inverse image of every compact subset of $M \times M$ under this map is a compact subset of $G \times M$. It can be shown that if $G$ is a compact Lie group then any smooth action of $G$ on manifolds is proper.

Proposition 8. A $G$-action on $M$ is proper if and only if for every two compact subsets $K_1$ and $K_2$ of $M$ the subset

$$\{g \in G : g K_1 \cap K_2 \neq \emptyset\}$$

is compact.

Unless there is the need to be more explicit, we write $\rho(g, p) = gp$. Define on $M$ the equivalence relation such that $p \sim p'$ if and only if $p' = gp$ for some $g \in G$. The set of equivalence relations is the quotient space, denoted $M/G$. Each element of the quotient space $M/G$ is then an orbit $Gp$ of a point $p$. We also refer to $M/G$ as the orbit space of the group action. The projection map $\pi : M \to M/G$ is by definition the maps that associates to each $p \in M$ the orbit $Gp$ in $M/G$.

Theorem 9. Let $G$ be a Lie group acting smoothly, properly, and freely on a manifold $M$. Then the quotient space $M/G$ admits a unique smooth manifold structure with the property that the projection map $\pi : M \to M/G$ is a smooth submersion. Moreover, $\dim(M/G) = \dim M - \dim G$. A map $F : M/G \to N$ is smooth if and only if $F \circ \pi : M \to N$ is smooth.
1. **Regular values. Tu, Problem 9.1, page 108.** Define \( f: \mathbb{R}^2 \to \mathbb{R} \) by

\[
f(x, y) = x^3 - 6xy + y^2.
\]

Find all values \( c \in \mathbb{R} \) for which the level set \( f^{-1}(c) \) is a regular submanifold of \( \mathbb{R}^2 \).

2. **Solution set of two equations. Tu, Problem 9.3, page 108.** Is the solution set of the system of equations

\[
x^3 + y^3 + z^3 = 1, \quad z = xy
\]

in \( \mathbb{R}^3 \) a smooth manifold? Prove your answer.

3. **(The special linear and orthogonal groups.)** The sets

\[
SL(n, \mathbb{R}) = \{ A \in M(n, \mathbb{R}) : \det A = 1 \}, \quad O(n) = \{ A \in M(n, \mathbb{R}) : A^t A = I \}
\]

are subgroups of the group \( GL(n, \mathbb{R}) \) of invertible real matrices of size \( n \)-by-\( n \). They are called, respectively, the *special linear* and the *orthogonal* groups. The intersection \( SO(n) = SL(n, \mathbb{R}) \cap O(n) \) is called the *special orthogonal group*.

(a) Show that \( SL(n, \mathbb{R}) \) is a Lie group. For this, first show that the differential of the determinant function

\[
det: M(n, \mathbb{R}) \to \mathbb{R}
\]

at any non-singular (i.e., invertible) matrix \( A \) is surjective.

(b) Show that \( O(n) = \{ A \in M(n, \mathbb{R}) : A^t A = I \} \) is a Lie group. For this, first show that the map \( f(A) = A^t A \) from \( M(n, \mathbb{R}) \) into the space of symmetric, positive, \( n \)-by-\( n \) matrices is a submersion at each \( A \in O(n) \).

(c) Explain that \( SO(n) = O(n) \cap SL(n, \mathbb{R}) \) is a Lie group.

4. **(The \( n \)-sphere \( S^n \))** The \( n \)-dimensional sphere is the set \( S^n = \{ q \in \mathbb{R}^{n+1} : \|q\| = 1 \} \).

(a) Show that \( S^n \) is a smooth submanifold of \( \mathbb{R}^{n+1} \). Do this by showing that for all non-zero \( q \) the differential at \( q \) of the map \( f: \mathbb{R}^{n+1} \to \mathbb{R} \) defined by \( f(q) = \|q\|^2 \) is surjective.

(b) Explain why the quotient \( SO(n+1)/SO(n) \) admits a smooth manifold structure with respect to which the projection map \( SO(n+1) \to SO(n+1)/SO(n) \) is smooth.

(c) The special orthogonal group \( SO(n+1) \) acts smoothly on the sphere \( S^n \) according to the action map \( (A, x) \to Ax \). Let \( \mathcal{N} = (0, \ldots, 0, 1)^t \) denote the north pole. We may regard \( SO(n) \) as the subgroup

\[
SO(n) = \{ A \in SO(n+1) : A\mathcal{N} = \mathcal{N} \}.
\]

Now define a map \( \varphi: SO(n+1)/SO(n) \to S^n \) by \( \varphi: gSO(n) \to g\mathcal{N} \). Show that this map is a diffeomorphism.

5. **(The torus \( T^n \)).** Let \( \mathbb{Z}^n \) be the subgroup of \( \mathbb{R}^n \) of integer vectors. Then \( \mathbb{Z}^n \) acts by translations of \( \mathbb{R}^n \). Define the \( n \)-torus as the quotient \( T^n = \mathbb{R}^n / \mathbb{Z}^n \). Explain why \( T^n \) is a smooth manifold and show that \( T^n \) is diffeomorphic to the \( n \)-fold product of circles \( S^1 \times \cdots \times S^1 \).

6. **(The real projective space \( P^n(\mathbb{R}) \)).** Let \( P^n(\mathbb{R}) \) denote the *real projective space*, which is defined as the quotient space of the \( n \)-sphere \( S^n \) under the relation: \( p \sim p' \) if and only if \( p' = \pm p \). Thus \( P^n(\mathbb{R}) = S^n / G \) where \( G \) is the two-element group \( \{ \pm I \} \). Explain that the real projective space has a unique smooth manifold structure in dimension \( n \) for which the projection \( \pi: S^n \to P^n(\mathbb{R}) \) is a smooth submersion.
7. **(The Möbius band)** Consider the cylinder \( C = \{ u = (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \} \). Define the action of \( G = \{ \pm 1 \} \) on \( C \) by \((g, u) = gu\). The quotient space \( M = C/(\pm 1) \) is called the **Möbius band**.

(a) Show that the Möbius band is a smooth manifold of dimension 2.

(b) Show that the Möbius band is diffeomorphic to \( P^2(\mathbb{R}) \) minus one point. (Note that the cylinder \( C \) is diffeomorphic to \( S^2 \) minus a pair of antipode points.)