1. **Partitions of Unity.** Read Section §13, Chapter 3 (pages140-147) of Tu’s book, concerning the construction of $C^\infty$ partitions of unity subordinated to an open cover of any given manifold.

2. **Existence of Riemannian metrics.** Let $M$ be a smooth manifold. A Riemannian metric on $M$ is the assignment, for each $p \in M$, of a real inner product $\langle \cdot, \cdot \rangle_p$ on $T_p M$. The Riemannian metric is said to be smooth (or $C^\infty$) if for all smooth vector fields $X, Y$ on $M$, the function $p \mapsto \langle X_p, Y_p \rangle_p$ is $C^\infty$. Often the Riemannian metric is denoted $g$, and the inner product at $p$ is written $g_p(u, v)$ for $u, v \in T_p M$.

Prove that every smooth manifold carries a smooth Riemannian metric. (In fact, many of them.) Suggestion: choose an atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ for $M$ and a partition of unity subordinate to the cover $\{U_\alpha : \alpha \in A\}$ of $M$. Now define a Riemannian metric locally, on each coordinate chart, and use the partition of unity to product another on the whole of $M$.

**Solution.** Let $\{\rho_\alpha : \alpha \in A\}$ be a smooth partition of unity subordinate to a covering $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of $M$ by coordinate neighborhoods. Recall that this means the following: $\mathcal{U}$ is a locally finite covering (meaning that every $p \in M$ admits a neighborhood $U$ such that $U \cap U_\alpha \neq \emptyset$ at most for a finite number of elements $U_\alpha$ of $\mathcal{U}$) and $\{\rho_\alpha : \alpha \in A\}$ is a family of smooth functions on $M$ for which the following conditions hold:

- $\rho_\alpha \geq 0$ and $\rho_\alpha = 0$ on the complement of the closure of $U_\alpha$, for each $\alpha \in A$.
- $\sum_\alpha \rho_\alpha(p) = 1$ for all $p \in M$.

Let $g_\alpha$ be a Riemannian metric on $U_\alpha$ for each $\alpha \in A$. For example, for each $p \in U_\alpha$ and $u, v \in T_p M$, let

$$g^\alpha_p(u, v) = \left(d\phi_\alpha^p u\right) \cdot \left(d\phi_\alpha^p v\right),$$

where $\phi_\alpha$ is the coordinate map on $U_\alpha$ and $\cdot$ denotes ordinary dot product in $\mathbb{R}^n$. Now define, for each $p \in M$ and $u, v \in T_p M$,

$$g_p(u, v) = \sum_{\alpha \in A} \rho_\alpha(p) g^\alpha_p(u, v).$$

Due the locally finite property of $\mathcal{U}$, the sum contains only finitely many terms for all $p$ in some neighborhood of each $p$, so $q \mapsto g_q(X_q, Y_q)$ is smooth for any pair of smooth vector fields $X, Y$ on $M$. Since the $\rho_\alpha$ are nonnegative and are not all equal to 0 for any given $p$, then $g_p$ is a positive definite symmetric bilinear form, thus a positive inner product. Therefore $g$ is a Riemannian metric on $M$. 

\[\Diamond\]
3. Vector fields and flows. Read Section §14, Chapter 3 (pages 149-160) of Tu’s book, concerning vector fields on manifolds, local and global flows, the Lie bracket of vector fields, and the push-forward operation. 

4. A vector field and its local flow in \( \mathbb{R} \). Consider the vector field on \( U = (0, \infty) \subset \mathbb{R} \) given by

\[
X(x) = \frac{1}{x} \frac{d}{dx}.
\]

Find the local flow associated to \( X \) and the maximal intervals of existence. Check that the defining properties of a local flow are satisfied for this example.

**Solution.** The differential equation corresponding to this vector field is \( \dot{x} = 1/x \), or \( x \dot{x} = 1 \). Written as a differential, \( x \, dx = dt \). Integrating from 0 to \( t \): \( x(t)^2 - x(0)^2 = 2t \), and since \( x(t) \) is positive, we find \( x(t) = \sqrt{x(0)^2 + t} \). This gives the local flow

\[
\Phi_t(x) = \sqrt{t + x^2}.
\]

The maximal domain is the set \( \Omega = \{(t, x) \in \mathbb{R}^2 : x > 0, t \geq -x^2 \} \). The flow property is satisfied:

\[
\Phi_0(x) = x \quad \text{and} \quad \Phi_t(\Phi_s(x)) = \sqrt{t + s + x^2} = \Phi_{t+s}(x).
\]

5. A vector field and its local flow in \( \mathbb{R}^2 \). Repeat the previous problem for the vector field on \( \mathbb{R}^2 \) defined by

\[
X(x, y) = -x \frac{\partial}{\partial x} + (y + x^2) \frac{\partial}{\partial y}.
\]

**Solution.** The system of ordinary differential equations associated to this vector field is

\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= y + x^2.
\end{align*}
\]

Then \( x(t) = x(0)e^{-t} \) and

\[
\dot{y} - y = x(0)^2e^{-2t}.
\]

Multiplying left and right-hand sides by the integrating factor \( e^{-t} \) gives

\[
\frac{d}{dt} (e^{-t} y) = x(0)^2 e^{-3t}.
\]

Integrating from 0 to \( t \) yields

\[
e^{-t} y(t) - y(0) = \frac{x(0)^2}{3} (1 - e^{-3t}).
\]

Therefore the solution of the system of equations is

\[
\begin{align*}
x(t) &= x(0)e^{-t} \\
y(t) &= y(0)e^t + \frac{1}{3} (e^t - e^{-2t}) x(0)^2.
\end{align*}
\]
The flow of $X$ is then

$$\Phi_t(x, y) = \left( e^{-tx}, e^t y + \frac{1}{3} (e^t - e^{-2t}) x^2 \right).$$

This is a global flow (i.e., it is defined for all $t \in \mathbb{R}$.) We now check the flow property. Clearly, $\Phi_0(x, y) = (x, y)$. Write $(x', y') = \Phi_t(x, y)$. Then

$$\Phi_s(\Phi_t(x, y)) = \left( e^{-s-t}x', e^{s-t} y' + \frac{1}{3} (e^{s-t} - e^{-2s}) (x')^2 \right)$$

$$= e^{-s-t}x, e^{s} \left( e^t y + \frac{1}{3} (e^t - e^{-2t}) x^2 \right) + \frac{1}{3} (e^s - e^{-2s}) e^{-2t} x^2$$

$$= \left( e^{-t}x, e^{s+t} y + \frac{1}{3} (e^{s+t} - e^{s+2t}) x^2 + \frac{1}{3} (e^{s+2t} - e^{-2(s+t)}) x^2 \right)$$

$$= \left( e^{-t}x, e^{s+t} y + \frac{1}{3} (e^{s+t} - e^{-2(s+t)}) x^2 \right)$$

$$= \Phi_{s+t}(x, y).$$

\[ \Box \]

6. **Left-invariant vector fields on the orthogonal group and their flows.** (This exercise requires knowledge of the basic facts about matrix exponentials.) Let $\mathfrak{o}(n) = \{ A \in M(n, \mathbb{R}) : A' = -A \}$. For each $A \in \mathfrak{o}(n)$ define the vector field on $M(n, \mathbb{R})$ expressed in the coordinates $x = (x_{ij})$ by

$$X_A(x) = \sum_{ij} (xA)_{ij} \frac{\partial}{\partial x_{ij}}.$$

(a) Show that $[X_A, X_B] = X_{[A,B]}$ for all $A, B \in \mathfrak{o}(n)$. Here $[A, B] = AB - BA$ is the matrix commutator.

(b) Show that the flow associated to $X_A$ is given by $\Phi_t(g) = g e^{tA}$ for all $g \in O(n)$.

(c) Show that $X_A(g) \in T_g O(n)$ for all $g \in O(n)$. For this, show that $dF_g X_A(g) = 0$, where $F(g) = g'g$.

(d) Show that $X_A$ is a left-invariant vector field on $O(n)$. For this, show that $(dL_g)_h X_A(h) = X_A(gh)$.

(e) Conclude that the Lie algebra of left-invariant vector fields on $O(n)$ is isomorphic to $\mathfrak{o}(n)$ with the commutator bracket of matrices.

**Solution.** (a) Let $\delta_{ij}$ denote the $(i, j)$-entry of the identity matrix. Note that

$$\frac{\partial}{\partial x_{ij}} (xA)_{rs} = \sum_r \frac{\partial}{\partial x_{ij}} (x_{rl} a_{lj}) = \delta_{ir} a_{js}.$$

Then

$$[X_A, X_B] = \sum_{r,s,l} \left( (xA)_{ij} \frac{\partial}{\partial x_{ij}} (xB)_{rs} - (xB)_{ij} \frac{\partial}{\partial x_{ij}} (xA)_{rs} \right) \frac{\partial}{\partial x_{rs}}$$

$$= \sum_{r,s,l} \left[ (xA)_{ij} \delta_{ir} b_{js} - (xB)_{ij} \delta_{ir} a_{js} \right] \frac{\partial}{\partial x_{rs}}$$

$$= \sum_{r,s} \left[ (xAB)_{rs} - (xBA)_{rs} \right] \frac{\partial}{\partial x_{rs}}$$

$$= \sum_{r,s} \left[ x(A,B)_{rs} \right] \frac{\partial}{\partial x_{rs}}$$

$$= X_{[A,B]}.$$
7. A diffeomorphism centralizing a flow. Let $g$ be a diffeomorphism of $U \subset \mathbb{R}^n$ and $\Phi^X_t$ a local flow on $U$ with infinitesimal generator $X$. If $g \ast X = X$ show that

$$g \circ \Phi^X_t = \Phi^X_t \circ g.$$ 

More generally, if $Y = g \ast X$ show that $\Phi^Y_t = g \circ \Phi^X_t \circ g^{-1}$ is a local flow with infinitesimal generator $Y$.

**Solution.** Let $\Psi_t := g \circ \Phi^X_t \circ g^{-1}$. Then

$$\frac{d}{dt} \Psi_t(p) = \frac{d}{dt} g(\Phi^X_t(g^{-1}(p))) = d g \Phi^X_t(g^{-1}(p)) \frac{d}{dt} \Phi^X_t(g^{-1}(p)) = d g \Phi^X_t(g^{-1}(p)) X(\Phi^X_t(g^{-1}(p))) = (g \ast X)(\Psi_t(p)).$$

Therefore,

$$\frac{d}{dt} \Psi_t(p) = Y(\Psi_t(p)).$$

By uniqueness of the flow, $\Phi^Y_t = g \circ \Phi^X_t \circ g^{-1}$.

8. Flows of commuting vector fields. If $X$ and $Y$ are commuting vector fields on a smooth manifold $M$ and $c \in \mathbb{R}$, show:

(a) $\Phi^cX_t = \Phi^X_{ct}$

(b) $\Phi^{X+Y}_t = \Phi^X_t \circ \Phi^Y_t$.

Thus we may think of the flow of a vector field $X$ as a kind of exponential $e^{tX}$. In fact, if

$$X(p) = \sum_{ij} a_{ij} x_j(p) \frac{\partial}{\partial x_i}$$

then

$$e^{tX}(p) = \exp\left(\sum_{ij} a_{ij} x_j(p) t \frac{\partial}{\partial x_i}\right).$$
10. More on commuting vector fields. Let $X_1, \ldots, X_n$ be everywhere linearly independent and commuting smooth vector fields defined on an open set $U \subset M$ of a smooth, $n$-dimensional manifold $M$. Let $\Phi_{X_i}^t$ be the local flow of $X_i$. We assume that $V \subset U$ is an open subset and $I = (-a, a)$ is a small enough interval such that $\Phi_{X_i}^t(p)$ is defined for all $p \in V$ and all $(t_1, \ldots, t_n) \in I^n$. For a fixed $p_0 \in V$ define

$$\varphi(t_1, \ldots, t_n) := \left(\Phi_{X_1}^{t_1} \circ \cdots \circ \Phi_{X_n}^{t_n}\right)(p_0).$$

Show that $\varphi$ is a diffeomorphism from $I^n$ onto its image in $V$ such that $\varphi_* \frac{\partial}{\partial t_i} = X_i$. Therefore the $X_i$ are the coordinate vector fields of the coordinate neighborhood of $p_0$ defined by the inverse of $\varphi$.

**Solution.** By a similar argument to that used in problem 8 we can show

$$\frac{\partial}{\partial t_j} \varphi(t_1, \ldots, t_n) = X_j(\varphi(t_1, \ldots, t_n)).$$

Stated differently, with $t = (t_1, \ldots, t_n)$, we have $d\varphi_t \frac{\partial}{\partial t_j} = X_j$. This implies (as the $X_j$ are linearly independent) that $d\varphi_t$ is an isomorphism from $T_p\mathbb{R}^n$ to $T_{\varphi(t)}M$. By the inverse function theorem, $\varphi$ is a local diffeomorphism. It also follows that in the parametrization defined by $\varphi$ the vector fields $X_j$ are the coordinate vector fields associated to this parametrization.

\[\diamond\]

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\[\diamond\]

10. Finding the Lie group for a given Lie algebra of vector fields. Let $\Phi_{X}^t$ and $\Phi_{Y}^t$ be global flows on a smooth manifold $M$ whose infinitesimal generators $X, Y$ satisfy the Lie bracket relation $[X, Y] = cY$, where $c$ is a real constant.

(a) Show that $\Phi_{X}^t \circ \Phi_{Y}^s = \Phi_{Y}^{-ct} \circ \Phi_{X}^t$.

(b) Give $\mathbb{R}^2$ the binary operation $(x_1, y_1)(x_2, y_2) := (x_1 + x_2, y_1 e^{c x_2} + y_2)$. Show that this defines a group multiplication on $\mathbb{R}^2$, making $\mathbb{R}^2$ a (non-abelian if $c \neq 0$) Lie group with identity $(0, 0)$ and inverse $(x, y)^{-1} = (-x, -ye^{-cx})$.

(c) Show that the map $(x, y) \mapsto \Phi_{X}^x \circ \Phi_{Y}^y$ defines a group isomorphism from $\mathbb{R}^2$, with the group multiplication given in part (b), to a two-dimensional subgroup of the group of diffeomorphisms of $M$ (with composition of maps as the group multiplication).
Solution.

(a) Writing the bracket relation as $\mathcal{L}_X Y = cY$ we have
\[
\frac{d}{dt} (\Phi_{s,t}^X) Y = \frac{d}{ds} \bigg|_{s=0} (\Phi_{s,t}^X) Y = \frac{d}{ds} \bigg|_{s=0} (\Phi_{s,t}) Y = \mathcal{L}_X (\Phi_{s,t}^X) Y = (\Phi_{s,t}^X)_s \mathcal{L}_X Y = c (\Phi_{s,t}^X)_s Y
\]
Thus, setting $Z(t) := (\Phi_{s,t}^X)_s Y$ gives the equation $Z' = cZ$, so $Z(t) = e^{ct} Z(0) = e^{ct} Y$. It follows that
\[
(\Phi_{t}^X)_t Y = e^{-ct} Y.
\]
From this we obtain
\[
\frac{d}{ds} \Phi_{t}^X \circ \Phi_{s,t}^Y(p) = ((\Phi_{t}^X)_t Y)(\Phi_{s,t}^X \circ \Phi_{s,t}^Y(p)) = e^{-ct} Y(\Phi_{s,t}^X \circ \Phi_{s,t}^Y(p)).
\]
On the other hand, by the chain rule,
\[
\frac{d}{ds} \Phi_{s,te^{ct}}^Y(p) = e^{-ct} Y(\Phi_{s,te^{ct}}^Y(p)).
\]
This shows that $s \mapsto \Phi_{s,te^{ct}}^Y \circ \Phi_{s,te^{ct}}^X(p)$ and $s \mapsto \Phi_{s,te^{ct}}^Y \circ \Phi_{s,te^{ct}}^X(p)$ are two curves satisfying the same initial value problem. (Both curves are at $p$ for $s = 0$ and satisfy the same differential equation.) By uniqueness they are the same, so $\Phi_{s,te^{ct}}^Y \circ \Phi_{s,te^{ct}}^X(p) = \Phi_{s,te^{ct}}^Y(p)$ as claimed.

(b) We need to check for associativity and the existence of an identity and inverse elements. These are straightforward computations. First, associativity:
\[
(x_0, y_0) \big[(x_1, y_1)(x_2, y_2)\big] = (x_0 + x_1 + x_2, y_0 e^{c(x_1+x_2)} + y_1 e^{c x_2} + y_2) = [(x_0, y_0)(x_1, y_1)](x_2, y_2).
\]
Then $(0,0)$ is the identity element:
\[
(x, y)(0,0) = (x + 0, ye^{0} + 0) = (x, y) = (0 + x, 0e^{ct} + y) = (0,0)(x, y).
\]
Finally, the group inverse is given by $(x, y)^{-1} = (-x, ye^{-cx})$ since
\[
(x, y)(-x, ye^{-cx}) = (x - x, ye^{-cx} - ye^{-cx}) = (0,0) = (-x + x, ye^{-cx} e^{cx} + y) = (-x, ye^{-cx})(x, y).
\]

(c) We first show that the map $(x, y) \mapsto \Phi_{x_2}^X \circ \Phi_{y_2}^Y$ is a homomorphism of groups:
\[
(x_1, y_1)(x_2, y_2) = (x_1 + x_2, y_1 e^{c y_2} + y_2) \mapsto \Phi_{x_1 + x_2}^X \circ \Phi_{y_1 e^{c y_2} + y_2}^Y = \Phi_{x_1}^X \circ \Phi_{x_2}^X \circ \Phi_{y_1 e^{c y_2} + y_2}^Y = \Phi_{x_1}^X \circ \Phi_{y_1}^Y \circ \Phi_{x_2}^X \circ \Phi_{y_2}^Y
\]
where, at the last step, we use the relation proved in part (a). Clearly $(0,0) \mapsto \Phi_0^X \circ \Phi_0^Y$ is the identity transformation. Therefore the homomorphism is in fact an isomorphism onto its image in the full group of diffeomorphisms of $\mathbb{R}^2$. 

\diamond