1. **Curves in 3-space.** Let \( c(t) = \left( t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3} \right) \), \( t \geq 0 \) be a parametric curve in \( \mathbb{R}^3 \).

(a) Find the length \( s(t) \) of \( c \) restricted to the interval \([0, t]\).

(b) Find the Frenet-Serret frame vectors \( T, N, B \).

(c) Find the curvature \( \kappa(t) \).

(d) Find the torsion \( \tau(t) \).

Remarks: The (positive) curvature \( \kappa \) and torsion \( \tau \) of a space curve are defined on page 15, Exercise 2.9 in the textbook. For clarity, I suggest writing \( \dot{c}(t) \) for the derivative in \( t \) and \( f'(s) \) for the derivative in the arc length \( s \). Thus, for example, if \( \gamma(s) \) represents the curve parametrized by arc length, then \( c(t) = \gamma(s(t)) \) and the unit tangent vector becomes \( T(t) = \gamma'(s(t)) = \dot{c}(t)/\dot{s}(t) \).

2. **Surface curvatures.** A surface of revolution in \( \mathbb{R}^3 \) is given the parametrization

\[
\Psi(u, v) = \begin{pmatrix}
a \cos v \\
a \sin v \\
b u
\end{pmatrix}
\]

where \( u > 0 \), \( 0 \leq v < 2\pi \) and \( a, b \) are positive constants such that \( a^2 + b^2 = 1 \). We assume the surface is oriented so that its normal vector field \( N \) points in the direction of \( \frac{\partial \Psi}{\partial u} \times \frac{\partial \Psi}{\partial v} \). Obtain in terms of the parameters \( u, v \) the values of

(a) The principal curvatures \( \kappa_1, \kappa_2 \).

(b) The mean curvature \( H \).

(c) The Gauss curvature \( K \).

3. **Conformally Euclidean Riemannian metrics.** Let \( \eta(x) = e^{\rho(x)} \) be a positive real valued smooth function defined on a non-empty open set \( D \subseteq \mathbb{R}^n \). We define on \( D \) a Riemannian metric as follows:

\[
\langle u, v \rangle_x = \eta(x)^2 u \cdot v
\]

where \( u, v \in T_x D = \mathbb{R}^n \) and \( u \cdot v \) is the ordinary dot-product in \( \mathbb{R}^n \).

(a) If \( x = x(t) \) is a geodesic path with respect to \( \langle \cdot, \cdot \rangle \), show that \( \eta(x)|\dot{x}| \) is constant in \( t \). (Here \( \dot{x} \) indicates the derivative in \( t \).)

(b) Show that the Levi-Civita connection for this metric satisfies

\[
\nabla_X Y = (X \rho) Y + (Y \rho) X - X \cdot Y \grad\rho.
\]
where $X, Y$ are constant (that is, parallel with respect to the Euclidean connection) vector fields on $\mathbb{D}$.

(c) Show that a smooth path $x(t) = (x_1(t), \ldots, x_n(t))$ is a geodesic for the conformally Euclidean metric if and only if it satisfies the system of differential equations

$$\ddot{x}_j + \sum_i \left( 2\dot{x}_i \dot{x}_j - |\dot{x}|^2 \delta_{ij} \right) \frac{\partial \rho}{\partial x_i} = 0, \quad j = 1, \ldots, n.$$ 

Here $\delta_{ij}$ are the entries of the identity matrix.

(d) Show that a vector field $X(t) = \sum_j h_j(t) \partial/\partial x_j$ is parallel along a curve $x(t) = (x_1(t), \ldots, x_n(t))$ if and only if the functions $h_j(t)$ satisfy the linear system of first order differential equations

$$\dot{h}_j + \sum_i \left\{ (h_j \dot{x}_i + h_i \dot{x}_j) - h \cdot \ddot{x}_i \delta_{ij} \right\} \frac{\partial \rho}{\partial x_i} = 0, \quad j = 1, \ldots, n.$$ 

(e) Let us introduce the parameter $\zeta(t)$ such that $d\zeta = \frac{1}{\eta(x)} \cdot \dot{x}$. We then define $z(\zeta) = x(t(\zeta))$. Show that $x(t)$ is a geodesic for $\langle \cdot, \cdot \rangle$ if and only if $z(\zeta)$ satisfies the differential equation

$$m\ddot{z} = -\nabla_{\ddot{z}} U$$ 

where $m = (\eta(x)|\dot{x}|)^{-2}$ is a constant of motion by part (a) and $U = E - \frac{1}{2} \eta^2$ for an arbitrary constant $E$. (Note: $\nabla_{\ddot{z}}$ above is the ordinary Euclidean gradient.)

Note that $\dot{x}$ indicates derivative in $t$, and $\ddot{x}$ indicates derivative in $\zeta$.

Remark: When $\rho \geq 0$, we may interpret $\eta(x)$ physically as the index of refraction of an optically transparent medium. Geodesics in the metric $\langle \cdot, \cdot \rangle$ are then paths traveled by a light ray. The above time change transforms the geodesic equation into Newton’s equation for a mechanical particle with mass $m$, with clock defined by the time parameter $\zeta$, and moving under the influence of a potential function $U = E - \frac{\eta^2}{2}$. Because $|\dot{z}| = |\dot{x}|\eta^2 = m^{-1/2} \eta$, it follows that

$$E = \frac{1}{2} m |\dot{z}|^2 + U$$

which allows us to interpret the constant $E$ as the total energy (kinetic plus potential) of a moving particle. (These sorts of ideas are associated to names like Euler, Fermat, Hamilton.)

4. **Riemannian geometry of a Lie group.** The Lie group $SL(2, \mathbb{R})$ is the matrix group consisting of the $2 \times 2$ real matrices with determinant 1. The Lie algebra of $SL(2, \mathbb{R})$ (regarded as the 3-dimensional vector space of left-invariant vector fields) is the linear span of left-invariant vector fields $E, F, H$ satisfying the Lie bracket relations


Let us define a Riemannian metric on $SL(2, \mathbb{R})$ by the sufficient requirement that $E, F, H$ constitute a global orthonormal frame of vector fields.

(a) If $V$ is the Levi-Civita (or Riemannian) connection, find

$$\nabla_E H, \nabla_F H, \nabla_H E, \nabla_H F, \nabla_E F, \nabla_E E, \nabla_F F, \nabla_H H.$$ 

(b) Find the sectional curvatures $K(EE) = \langle R(E, F)F, E \rangle$, $K(EH) = \langle R(E, H)H, E \rangle$, $K(FH) = \langle R(F, H)H, F \rangle$.

5. **The Hopf bundle and Chern number.** This exercise begins with a long description of some preliminary information. We wish to define a complex line bundle (that is to say, a vector bundle over $\mathbb{C}$ of rank one) over
the sphere $S^2$, and compute its Chern number. (See pages 234 and 235 of the textbook for the basic information about complex vector bundles, Hermitian metrics, and Chern classes.) We haven't had the opportunity to actually compute characteristic classes of a vector bundle in a concrete example in class, so this is the last opportunity for us to finally do so. The actual exercise, beyond reading this lengthy preamble, won't itself be too long.

It will be convenient throughout to use complex numbers. In particular, we will use the fact that the unit sphere $S^2$ is diffeomorphic to the complex projective space $CP^1$. This space is defined as the set of complex lines (one-dimensional complex vector subspaces) in $C^2$. Formally, it is defined as the quotient of $C^2 \setminus \{0\}$ (the 2-dimensional complex vector space minus the origin) under the equivalence relation that identifies any two non-zero vectors that are collinear. Equivalently, let $S^3$ denote the 3-dimensional unit sphere, regarded as the submanifold of $C^2$ consisting of pairs $(z_1, z_2)$ such that $|z_1|^2 + |z_2|^2 = 1$, and define on it the action by the group $U(1)$ of unit modulus complex numbers:

$$\left(e^{i\theta}, (z_1, z_2)\right) \mapsto \left(e^{i\theta}z_1, e^{i\theta}z_2\right).$$

Then $CP^1$ is the quotient of $S^3$ under this action. Notice that the orbits are circles, so we should expect the quotient to be a 2-dimensional (real) manifold. In fact, $CP^1$ has the structure of a smooth manifold diffeomorphic to $S^2$. (It is also a 1-dimensional complex manifold.) The quotient map defines a circle bundle $\pi: S^3 \to S^2$ called the Hopf bundle.

⋄

You may skip this part till the next diamond (⋄) sign. The following comments are inspired by a view from quantum theory that may help see the Hopf bundle somewhat differently and help to convince ourselves without a detailed proof that $S^2 \cong CP^1$. An alternative definition of $CP^1$ will be useful. Let $M$ denote the space of self-adjoint $2 \times 2$ complex matrices of trace 0. Such matrices can be written as

$$\sigma \cdot x := x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}$$

where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the so called Pauli matrices. We are interested in $x \in S^2$. Note that

$$(\sigma \cdot x)^2 = |x|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

where $|x|^2 = x_1^2 + x_2^2 + x_3^2 = 1$ and $I$ is the identity matrix. The matrices $\sigma \cdot x$ for $x \in S^2$ have eigenvalues $\pm 1$. In fact, consider the matrices $P_\pm(x) = \frac{1}{2}(I \pm \sigma \cdot x)$. They are the orthogonal projections to the 1-dimensional subspaces of $C^2$ spanned by the eigenvalues $\pm 1$. This is a consequence of the easy to verify matrix identities (where $P^\ast$ indicates the transpose-conjugate, or matrix adjoint):

$$P_\pm(x)^* = P_\pm(x), \quad P_\pm(x)^2 = P_\pm(x), \quad P_+(x)P_-(x) = P_+(x)P_-(x) = 0, \quad P_-(x) + P_+(x) = I, \quad \sigma \cdot x P_\pm(x) = \pm P_\pm(x).$$

We have in this way used $S^2$ to parametrize all the (simultaneously) self-adjoint and unitary $2 \times 2$ complex matrices having distinct eigenvalues. Let $u(x)$ be a unit length eigenvector in $C^2$ associated to the eigenvalue 1.
Thus \( u(x) \in S^3 \) and \( P_\ast(x)u(x) = u(x) \). Observe that \( P_\ast(x) \) is the rank-1 orthogonal projection matrix given by 
\[
   u(x) \otimes u(x)^* \text{, where here } u(x)^* \text{ is the dual vector: } u(x)^* = \langle u(x), \cdot \rangle \text{ and } \langle \cdot, \cdot \rangle \text{ is the Hermitian inner product in } \mathbb{C}^2 \text{ given by: } \langle (z_1, z_2), (w_1, w_2) \rangle = z_1 w_1 + \overline{z}_2 w_2. \]
In other words, for any \( v \in \mathbb{C}^2 \),
\[
   P_\ast(x)v = \langle u(x), v \rangle u(x).
\]
The outcome of the above discussion is this: if we identify each element of projective space \( CP^1 \) with the orthogonal projection operator \( P_\ast(x) \) in the Hilbert space \( \mathbb{C}^2 \) onto the eigenspace of \( \sigma \cdot x \) for eigenvalue 1, we obtain a map \( x \in S^2 \rightarrow u(x) \otimes u(x)^* \in CP^1 \) which we can prove to be a diffeomorphism. (In this way, the complex projective space can be regarded as a submanifold of a three-dimensional (over \( \mathbb{R} \) space of matrices.) Note that this map is well-defined since a different choice \( e^{i\theta} u(x) \) of unit length eigenvector will give the same projection map \( u(x) \otimes u(x)^* \). On the other hand, each projection can be written as the image of a point in \( S^3 \) under the map
\[
   \pi : S^3 = \{ z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \} \rightarrow CP^1 \cong S^2
\]
such that \( \pi(z) = z \otimes z^* \). Observe that \( \pi(e^{i\lambda} z) = \pi(z) \) since
\[
   \pi(e^{i\lambda} z) = (e^{i\lambda} z) \otimes (e^{i\lambda} z)^* = e^{i\lambda} e^{-i\lambda} z \otimes z^* = z \otimes z^* = \pi(z).
\]
So we have, as expected, that \( CP^1 \) is a quotient of \( S^3 \) under the diagonal action of the circle group \( U(1) \) on \( S^3 \):
\[
   e^{i\lambda}(z_1, z_2) = (e^{i\lambda} z_1, e^{i\lambda} z_2).
\]
The identification between \( S^2 \) and \( CP^1 \) is accomplished by the bijection \( x \rightarrow \sigma \cdot x \). The resulting map \( \pi : S^3 \rightarrow S^2 \) is, again, the Hopf bundle.
\[
   \circ
\]
In what follows, elements of \( CP^1 \) will be written as \( [z] = [z_1 : z_2] \) (so-called homogeneous coordinates) to indicate the equivalence class represented by \( z \in \mathbb{C}^2 \setminus \{(0, 0)\} \) under the action of the multiplicative group of non-zero complex numbers. (Or, equivalently, \( z \in S^3 \) under the action of \( U(1) \).)
The Hopf bundle is the bundle of unit length vectors of a complex line bundle over \( CP^1 \) called the tautological bundle, which is a rank 1 (complex) subbundle \( L \) of the trivial bundle \( \text{pr}_1 : CP^1 \times \mathbb{C}^2 \rightarrow CP^1 \) defined by the property that the fiber \( L_{[z]} \) of \( L \) at \( [z] \) the one-dimensional complex subspace in \( \mathbb{C}^2 \) spanned by \( z \) itself (thus the name tautological). I’ll use the same letter \( \pi \) to denote the base point map \( \pi : L \rightarrow CP^1 \). By definition,
\[
   L = \{[\ell, z] : [z] \in CP^1, \lambda \in \mathbb{C} = \{[\ell, z] \in CP^1 \times \mathbb{C}^2 : z \in \ell \}, \pi(\ell, z) = \ell.
\]
The trivial vector bundle \( CP^1 \times \mathbb{C}^2 \) can be endowed with a Hermitian metric by giving \( \mathbb{C}^2 \) the standard (complex valued) inner product
\[
   \langle (z_1, z_2), (w_1, w_2) \rangle = \overline{z}_1 w_1 + \overline{z}_2 w_2.
\]
(Note: as a matter of taste, I deviate from the textbook convention by placing the complex conjugate bar over the first vector argument of the inner product.) The Hopf bundle is then the bundle of unit length vectors of \( L \). (It is a principal bundle over \( CP^1 \) with structure group \( U(1) \), according to the definitions at the beginning of Chapter 6 of the textbook, although we don’t need this fact for this assignment.) We thus have \( S^3 \subseteq L \subseteq CP^1 \times \mathbb{C}^2 \), and these are all bundles over projective space.
Our goal is to compute the Chern number of \( L \), which is the integral over \( S^2 \equiv CP^1 \) of the Chern class \( c_1(L) \)
In fact, \( \Omega \) we will need the connection and curvature forms of \( \Pi \). Let 
\[ \Pi \vert_{\{z\}} : \mathcal{C}^2 \to \mathcal{L}_{\{z\}} \]
be the resulting projection map. We now define a connection \( \nabla \) on smooth sections of \( \mathcal{L} \) as follows: given \( \xi \in \Gamma(\mathcal{L}) \) and \( u \in T_{\{z\}} \mathbb{C}P^1 \),
\[
\nabla_u \xi = \Pi \vert_{\{z\}} D_u \xi
\]
where \( D \) is the ordinary derivative of a \( \mathcal{C}^2 \)-valued (equivalently, \( \mathbb{R}^4 \)-valued) function on \( \mathbb{C}P^1 \):
\[
D_u(f_1 + i f_2, f_3 + i f_4) = (u f_1 + i u f_2, u f_3 + u f_4).
\]

We will need the connection and curvature forms of \( \nabla \) for convenient choices of local trivializations of \( \mathcal{L} \). But first note that if \( \xi : U \to S^3 \subseteq L \) is a section of \( \mathcal{L} \) over \( U \subseteq \mathbb{C}P^1 \), then the corresponding connection form is \( \omega_p(u) = \langle \xi(p), \nabla_u \xi \rangle = \langle \xi(p), D_u \xi \rangle \) for \( u \in T_p \mathbb{C}P^1 \). This being an actual 1-form on \( U \), the associated curvature form is simply \( \Omega = d \omega + \omega \wedge \omega = d \omega \). The effect of taking sections of \( \mathcal{L} \) of unit length (hence in \( S^3 \)) is that \( \text{Re}(\langle \xi, D_u \xi \rangle) = 0 \). In fact,
\[
0 = u(\xi, \xi) = \langle \nabla_u \xi, \xi \rangle + \langle \xi, \nabla_u \xi \rangle = \langle \xi, \nabla_u \xi \rangle + \overline{\langle \xi, \nabla_u \xi \rangle} = 2 \text{Re}(\langle \xi, \nabla_u \xi \rangle).
\]

Therefore
\[
\omega_p(u) = 2 i \text{Im}(\langle \xi(p), D_u \xi \rangle).
\]

Observe that if \( \eta \) is another section also defined on \( U \), then \( \eta = f \xi \) where \( f \) is a complex-valued function on \( U \) of unit modulus: \( |f(p)| = 1 \). Then \( \omega_\eta = \omega_\xi + f^{-1} d f \) and \( d \omega_\eta = d \omega_\xi \). This means that the curvature form \( \Omega \) will define a global closed form on \( \mathbb{C}P^1 \). The Chern class is then (according to the definition on page 235 of the textbook in the special case of rank 1) \( c_1(L) = \frac{1}{2 \pi i} \Omega \) and the Chern number, our ultimate goal here, is the value of \( \int_{\mathbb{C}P^1} c_1(L) \).

We now choose trivializing neighborhoods and sections of the Hopf bundle. Let \( \{U_-, U_+\} \) be the open cover of \( \mathbb{C}P^1 \) given by
\[
U_- := \{|z_1 : z_2| \in \mathbb{C}P^1 : z_2 \neq 0\}, \quad U_+ := \{|z_1 : z_2| \in \mathbb{C}P^1 : z_1 \neq 0\}.
\]

We define coordinates \( \varphi \) : \( \mathbb{C} \to U_- \) by \( \varphi(z_1 : z_2) = z_- := z_1/z_2 \) and \( \varphi(z_1 : z_2) = z_+ := z_2/z_1 \). (These give us a smooth (in fact, holomorphic) atlas on \( \mathbb{C}P^1 \).) On \( U_+ \) we define the section \( \xi_+ \) as follows:
\[
\xi_-([z_1 : z_2]) := \frac{1}{\sqrt{1 + |z_2|^2}} \begin{pmatrix} z_- \\ 1 \end{pmatrix}, \quad \xi_+([z_1 : z_2]) := \frac{1}{\sqrt{1 + |z_2|^2}} \begin{pmatrix} 1 \\ z_+ \end{pmatrix}
\]

Associated to these sections we have connection forms \( \omega_- , \omega_+ \). You will show in one of the exercises below that on \( U_- \cap U_+ \)
\[
\omega_+ = \omega_+ + \overline{f} d f
\]
where \( f([z_1 : z_2]) = z_+/|z_+| \).

Before moving forward, let us try to get a better sense of what parts of \( S^2 \) are covered by \( U_+ \). Let us define the
map \( F : \mathbb{C}P^1 \to S^2 \) such that \( F([z_1 : z_2]) = (x_1, x_2, x_3) \) where

\[
x_1 = 2\text{Re} \left( z_1 \overline{z}_2 \right), \quad x_2 = 2\text{Im} \left( z_1 \overline{z}_2 \right), \quad x_3 = |z_1|^2 - |z_2|^2.
\]

We are assuming here that \(|z_1|^2 + |z_2|^2 = 1\). It is not difficult to see from this expression that \( F \) is injective. To see that it is surjective, observe that

\[
F \left( \left[ \cos \left( \frac{\varphi}{2} \right) e^{\frac{\psi}{2} i}, \sin \left( \frac{\varphi}{2} \right) e^{\frac{\psi}{2} i} \right] \right) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi).
\]

On the right-hand side we have an arbitrary point on \( S^2 \) expressed in spherical coordinates by allowing \( \theta \in [0, 2\pi) \) and \( \varphi \in [0, \pi] \). Changing \( \psi \) does not change the point \([z_1 : z_2] \) (\( \psi \) parametrizes the \( U(1) \) fiber of the Hopf bundle). From this description we see that \([1 : 0]\) is mapped to the North Pole \( N = (0, 0, 1)^T \) and \([0 : 1]\) to the South Pole \( S = (0, 0, -1)^T \), while \([1 : \frac{\psi}{\sqrt{2}} : \frac{\varphi}{\sqrt{2}}]\) is mapped to the equator \((x_3 = 0)\). Also note that the function \( f \) in (4) maps \([1 : \frac{\psi}{\sqrt{2}} : \frac{\varphi}{\sqrt{2}}]\) to \( e^{i\theta} \). We have that \((U_+, \varphi_+)\) parametrizes a region of \( S^2 \) that contains the northern hemisphere \( S^2_+ \) while \((U_-, \varphi_-)\) parametrizes a region containing the southern hemisphere \( S^2_- \). As \( \theta \) increases in \([1 : \frac{\psi}{\sqrt{2}} : \frac{\varphi}{\sqrt{2}}]\) we traverse the equation in the positive direction if we regard the equation as the boundary of the northern hemisphere, and in the negative direction relative to the boundary orientation of the southern hemisphere.

Also observe that \( f \left( \left[ \frac{1}{\sqrt{2}} : \frac{\psi}{\sqrt{2}} \right] \right) = e^{i\theta} \) and that the pull-back (or restriction) of \( f \) to the equator is

\[
e^{-i\theta} d\left( e^{i\theta} \right) = e^{-i\theta} e^{i\theta} i d\theta = i d\theta.
\]

Finally observe that

\[
\int_{\mathbb{C}P^1} \Omega = \int_{S^2_+} \omega_+ + \int_{S^2_-} \omega_- = \int_{\partial S^2_+} \omega_+ + \int_{\partial S^2_-} \omega_- = \int_{\partial S^2} (\omega_+ - \omega_-) = \int_0^{2\pi} i d\theta = i 2\pi.
\]

We conclude that

\[
\int_{\mathbb{C}P^1} c_1(L) = \int_{\mathbb{C}P^1} \frac{i \Omega}{2\pi} = \frac{i}{2\pi} 2\pi i = -1.
\]

Therefore the Chern number of the tautological line bundle over \( \mathbb{C}P^1 \) is \(-1\).

\( \diamond \)

(a) Let \( L^n \) be the line bundle over \( \mathbb{C}P^1 \) given by the tensor product \( L \otimes \cdots \otimes L \) \( (n \text{ times}) \). Show that the Chern number of \( L^n \) is \(-n\). (As a lemma, show that if \( L_1, L_2 \) are two line bundles then the first Chern class of their tensor product is \( c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \).)

(b) Let \( L^* \) denote the dual bundle to the tautological line bundle \( L \). Show that \( c_1(L^*) = -c_1(L) \). In particular, \( L^* \) has Chern number 1. (Note: \( L \otimes L^* \) is the bundle of endomorphisms of \( L \). This line bundle has a nowhere vanishing section given by the identity map on each fiber of \( L \).)

Knowing that the trivial line bundle has Chern number 0, we conclude from the above that every integer \( n \in \mathbb{Z} \) is the Chern number of some complex line bundle over \( \mathbb{C}P^1 \) (and line bundles with different Chern numbers are not isomorphic).