Homework set 3 - due 10/16/20

Math 5047 – Renato Feres

   (40 points.) A diffeomorphism \( \varphi : M \rightarrow \tilde{M} \) of smooth manifolds induces an isomorphism
   \[ \varphi_* : \mathfrak{X}(M) \rightarrow \mathfrak{X}(\tilde{M}) \]
   of their Lie algebras of vector fields. Recall that this means the following. If \( X \) is a smooth vector field on \( M \), then \( \varphi_* X \) is the vector field on \( \tilde{M} \) that can be defined by its action on functions \( f : \tilde{M} \rightarrow \mathbb{R} \) this way:
   \[ (\varphi_* X)_p f = X_{\varphi^{-1}(p)} (f \circ \varphi) \]
   for all \( p \in \tilde{M} \). The vector space \( \mathfrak{X}(M) \) is a Lie algebra under the Lie bracket operation. Then \( \varphi_* \) is a linear isomorphism such that
   \[ \varphi_* [X, Y] = [\varphi_* X, \varphi_* Y] \]
   for all \( X, Y \in \mathfrak{X}(M) \). Suppose \( \tilde{M} \) has an affine connection \( \tilde{\nabla} \). For \( X, Y \in \mathfrak{X}(M) \), define the vector field \( \nabla_X Y \in \mathfrak{X}(M) \) by
   \[ \nabla_X Y = (\varphi_*)^{-1} (\tilde{\nabla}_{\varphi_* X} (\varphi_* Y)) \].

   (a) Show that for all \( f \in C^\infty(M) \) we have \( \varphi_* (f X) = (f \circ \varphi X) \).
   (b) Show that for all \( X, Y \in \mathfrak{X}(M) \) we have \( \varphi_* [X, Y] = [\varphi_* X, \varphi_* Y] \).
   (c) Show that \( \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) \) is an affine connection.
   (d) If \( R \) is the curvature tensor of \( \nabla \) and \( \tilde{R} \) is the curvature tensor of \( \tilde{\nabla} \), show that
   \[ \varphi_* (R(X, Y) Z) = \tilde{R}(\varphi_* X, \varphi_* Y) \varphi_* Z \]
   for all \( X, Y, Z \in \mathfrak{X}(M) \).
   (e) If \( \varphi \) is an isometry of Riemannian manifolds and \( \tilde{\nabla} \) is the Levi-Civita connection on \( \tilde{M} \) (i.e., the Riemannian connection in the textbook terminology), then \( \nabla \) is the Levi-Civita connection on \( M \).

2. Symmetries of the curvature tensor. (20 points.) Let \( R \) be the curvature tensor of a Riemannian manifold \( M \). Show the following algebraic properties of \( R \) hold, where \( X, Y, Z, W \in \mathfrak{X}(M) \).

   (a) \[ \langle R(X, Y) Z, W \rangle = -\langle R(Y, X) Z, W \rangle \]
   (b) \[ \langle R(X, Y) Z, W \rangle = -\langle R(X, Y) W, Z \rangle \]
   (c) \[ R(X, Y) Z + R(Z, X) Y + R(Y, Z) X = 0 \] (Recall the Jacobi identity: \( [X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \).)
3. Geometry of tubes in \( \mathbb{R}^3 \). Let \( c(s) \) be a smooth closed curve in \( \mathbb{R}^2 \) parametrized by arc-length, where \( \mathbb{R}^2 \) is regarded as a plane in \( \mathbb{R}^3 \). Let \( e_1(s) := c'(s) \) (the tangent vector to the curve) and \( e_2(s) \) the unit orthogonal vector to \( e_1(s) \) so that \( e_3 := e_1(s) \times e_2(s) \) is the standard basis vector \((0, 0, 1)\) of \( \mathbb{R}^3 \). Define the parametric surface \( M \subseteq \mathbb{R}^3 \) with parametrization

\[
\Phi(s, \varphi) = c(s) + r \left[ \cos(\varphi)e_2(s) + \sin(\varphi)e_3 \right]
\]

for \( 0 \leq \varphi \leq 2\pi \). Notice that the image of \( \Phi \) is a tube of radius \( r \) with central (plane) curve \( c(s) \). Let \( \kappa(s) \) denote the (signed) curvature of \( c \) (textbook, page 11), so that

\[
e'_1(s) = \kappa(s)e_2(s), \quad e'_2(s) = -\kappa(s)e_1(s).
\]

We further define the vector fields \( E_1, E_2 \) on the parametric surface \( M \) by

\[
E_1(s, \varphi) = \frac{\partial \Phi}{\partial s}(s, \varphi), \quad E_2(s, \varphi) = \frac{\partial \Phi}{\partial \varphi}(s, \varphi), \quad N(s, \varphi) = \cos(\varphi)e_2(s) + \sin(\varphi)e_3.
\]

(a) Show that \( E_1, E_2 \) constitute an orthonormal frame on \( TM \) (i.e., an orthonormal basis of \( T_pM \) at each point \( p = \Phi(s, \varphi) \)), and that \( N \) is a unit normal vector field to \( M \).

(b) Show that the shape operator \( L \) of the tube satisfies at each point of \( M \):

\[
L(E_1(s, \varphi)) = \frac{\kappa(s)\cos(\varphi)}{1 - \kappa(s)\cos(\varphi)} E_1(s, \varphi), \quad L(E_2(s, \varphi)) = -\frac{1}{r} E_2(s, \varphi).
\]

(c) What is the Gaussian curvature of the parametric surface at each point \( p = \Phi(s, \varphi) \)?

(d) Let \( \nabla \) denote the Levi-Civita connection on \( M \). Obtain the vector fields:

\[
\nabla_{E_1} E_1, \quad \nabla_{E_2} E_2, \quad \nabla_{E_3} E_1, \quad \nabla_{E_3} E_2.
\]

4. The gradient and Hessian of a function. (20 points.) Let \( M \) be a Riemannian manifold and \( f : M \to \mathbb{R} \) a smooth function. We define the gradient of \( f \) as the vector field \( \text{grad} f \) such that, for every other vector field \( Z, p \in M \) and \( \nu \in T_pM \),

\[
\langle (\text{grad} f)(p), \nu \rangle = \nu f.
\]

Note that \( \nu f = df(\nu) \) is the directional derivative of \( f \) along \( \nu \). We define the Hessian of \( f \) at \( p \) as the quadratic
form on $T_p M$ given by

$$\text{Hess}(f)_p(u, v) := \langle \nabla_{u \text{grad}} f, v \rangle$$

where $\nabla$ is the Levi-Civita connection on $M$. Show that the Hessian is a symmetric quadratic form: $\text{Hess}_p(u, v) = \text{Hess}_p(v, u)$. (Note: we may define the Laplacian of the Riemannian manifold $M$ as the trace of the Hessian, $\text{Tr}(\text{Hess}(f))$.)