1. **Problems 8.1 and 8.2, page 64 of textbook: Transformation of an affine connection under a diffeomorphism.**

   (40 points.) A diffeomorphism \( \varphi : M \to \tilde{M} \) of smooth manifolds induces an isomorphism

   \[ \varphi_* : \mathcal{X}(M) \to \mathcal{X}(\tilde{M}) \]

   of their Lie algebras of vector fields. Recall that this means the following. If \( X \) is a smooth vector field on \( M \), then \( \varphi_* X \) is the vector field on \( \tilde{M} \) that can be defined by its action on functions \( f : \tilde{M} \to \mathbb{R} \) this way:

   \[ (\varphi_* X)_p f = X_{\varphi^{-1}(p)}(f \circ \varphi) \]

   for all \( p \in \tilde{M} \). The vector space \( \mathcal{X}(M) \) is a Lie algebra under the Lie bracket operation. Then \( \varphi_* \) is a linear isomorphism such that

   \[ \varphi_* [X,Y] = [\varphi_* X, \varphi_* Y] \]

   for all \( X, Y \in \mathcal{X}(M) \). Suppose \( \tilde{M} \) has an affine connection \( \tilde{\nabla} \). For \( X, Y \in \mathcal{X}(M) \), define the vector field \( \nabla_X Y \in \mathcal{X}(M) \) by

   \[ \nabla_X Y = (\varphi_*)^{-1}(\tilde{\nabla}_{\varphi_* X}(\varphi_* Y)) \]

   (a) Show that for all \( f \in C^\infty(M) \) we have \( \varphi_* (f X) = (f \circ \varphi^{-1}) \varphi_* X \).

   (b) Show that for all \( X, Y \in \mathcal{X}(M) \) we have \( \varphi_* [X,Y] = [\varphi_* X, \varphi_* Y] \).

   (c) Show that \( \nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \) is an affine connection.

   (d) If \( R \) is the curvature tensor of \( \nabla \) and \( \tilde{R} \) is the curvature tensor of \( \tilde{\nabla} \), show that

   \[ \varphi_* (R(X,Y)Z) = \tilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z \]

   for all \( X, Y, Z \in \mathcal{X}(M) \).

   (e) If \( \varphi \) is an isometry of Riemannian manifolds and \( \tilde{\nabla} \) is the Levi-Civita connection on \( \tilde{M} \) (i.e., the Riemannian connection in the textbook terminology), then \( \nabla \) is the Levi-Civita connection on \( M \).

**Solution.**

(a) Notice that this definition of \( \varphi_* X \) may be written as \( (\varphi_* X)h = [X(h \circ \varphi)] \circ \varphi^{-1} \). Denoting by \( U \) the composition operator \( Uh = h \circ \varphi \), then \( \varphi_* X = U^{-1}XU \). Now \( fX \) is the composition of the derivative operator \( X \) on \( C^\infty(M) \) and the multiplication operator \( M_fU = fU \). Note that \( M_fU = UU^{-1}fU^{-1}XU = f \circ \varphi^{-1}\varphi_* X \).
(b) We have
\[(\varphi_\ast X)(\varphi_\ast Y) = (U^{-1}XU)(U^{-1}YU) = U^{-1}XYU.\]

Thus
\[[\varphi_\ast X, \varphi_\ast Y] = (\varphi_\ast X)(\varphi_\ast Y) - (\varphi_\ast Y)(\varphi_\ast X) = U^{-1}(XY - YX)U = \varphi_\ast [X, Y].\]

(c) We need to check (i) $C^\infty(M)$-linearity in $X$ and (ii) the Leibniz property in $Y$.

   i. Note that $(\varphi_\ast)^{-1} = (\varphi^{-1})_\ast$. Then
   \[
   \nabla_{fX}Y = (\varphi_\ast)^{-1} (\nabla_{\varphi_\ast X}(\varphi_\ast Y))
   = (\varphi_\ast)^{-1} (\nabla_{\varphi_\ast X}(f(\varphi_\ast Y))
   = (\varphi_\ast)^{-1} \left( (f \circ \varphi^{-1}) \nabla_{\varphi_\ast X}(\varphi_\ast Y) \right)
   = f(\varphi_\ast)^{-1} (\nabla_{\varphi_\ast X}(\varphi_\ast Y))
   = f\nabla_XY.
   \]

   ii. For the Leibniz property, note:
   \[
   \nabla_X(fY) = (\varphi_\ast)^{-1} (\nabla_{\varphi_\ast X}(\varphi_\ast (fY)))
   = (\varphi_\ast)^{-1} (\nabla_{\varphi_\ast X}(f(\varphi_\ast (fY))))
   = (\varphi_\ast)^{-1} \left( ((f \circ \varphi^{-1}) \nabla_{\varphi_\ast X}(\varphi_\ast Y)) + f \circ \varphi^{-1} \nabla_{\varphi_\ast X}(\varphi_\ast Y) \right)
   = (\varphi_\ast)^{-1} \left( ((Xf) \circ \varphi^{-1}) (\varphi_\ast Y) + f \circ \varphi^{-1} \nabla_{\varphi_\ast X}(\varphi_\ast Y) \right)
   = (Xf)Y + f(\varphi_\ast)^{-1} \nabla_{\varphi_\ast X}(\varphi_\ast Y)
   = (Xf)Y + Y\nabla_XY.
   \]

(d) First notice that
\[
\nabla_X \nabla_Y Z = (\varphi_\ast)^{-1} \nabla_{\varphi_\ast X}(\varphi_\ast) - (\varphi_\ast) \nabla_{\varphi_\ast X}(\varphi_\ast) \nabla_{\varphi_\ast Y} \varphi_\ast Z = (\varphi_\ast)^{-1} \nabla_{\varphi_\ast X}(\varphi_\ast) \nabla_{\varphi_\ast Y} \varphi_\ast Z
\]

and
\[
\nabla_{[X,Y]} Z = (\varphi_\ast)^{-1} \nabla_{\varphi_\ast [X,Y]} \varphi_\ast Z = (\varphi_\ast)^{-1} \nabla_{\varphi_\ast [X,Y]} \varphi_\ast Z.
\]

So
\[
R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z
= (\varphi_\ast)^{-1} \nabla_{\varphi_\ast X}(\varphi_\ast) \nabla_{\varphi_\ast Y} \varphi_\ast Z - (\varphi_\ast)^{-1} \nabla_{\varphi_\ast Y}(\varphi_\ast) \nabla_{\varphi_\ast X}(\varphi_\ast) Z - (\varphi_\ast)^{-1} \nabla_{\varphi_\ast [X,Y]} \varphi_\ast Z
= (\varphi_\ast)^{-1} \tilde{R}(\varphi_\ast X, \varphi_\ast Y)(\varphi_\ast) Z.
\]

(e) We need to check that $\nabla$ is a metric connection and its torsion is 0. First recall that the condition of $\varphi$ being isometric means that for all $X, Y \in \mathcal{X}(M)$,
\[
\langle X, Y \rangle = (\varphi_\ast X)(\varphi_\ast Y) \circ \varphi.
\]

Writing $f = \langle X, Y \rangle$, also note that the definition of $\varphi_\ast Z$ implies
\[
Z(f \circ \varphi) = [\varphi_\ast Z] \circ \varphi.
\]
2. Symmetries of the curvature tensor. (20 points.) Let $R$ be the curvature tensor of a Riemannian manifold $M$. Show the following algebraic properties of $R$

(a) $(R(X, Y)Z, W) = - (R(Y, X)Z, W)$

(b) $(R(X, Y)Z, W) = - (R(X, Y)W, Z)$

(c) $R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$ (Recall the Jacobi identity: $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$)

(d) Let $V \subset T_pM$ be a 2-dimensional subspace and $\{e_1, e_2\}$ be a basis of $V$. Let $\{f_1, f_2\}$ be another basis of $V$ and $A = (a_{ij})$ the change of basis matrix (so that $f_i = \sum_j a_{ij} e_j$.) Show that

$$\langle R_p(f_1, f_2) f_2, f_1 \rangle = (\det A)^2 \langle R(e_1, e_2) e_2, e_1 \rangle.$$

Conclude that the number

$$K_p(V) := \langle R(e_1, e_2) e_2, e_1 \rangle,$$

defined in terms of an orthonormal basis $\{e_1, e_2\}$, does not depend on the choice of the orthonormal basis. $K$ is called the sectional curvature of $M$ at $p$ along $V$.

(e) Compute the sectional curvature of the sphere $S^{n-1}(R) = \{x \in \mathbb{R}^n : \|x\| = R\}$ of radius $R$ on an arbitrary 2-dimensional subspace of any tangent space of the sphere. (Suggestion: Use the Gauss curvature equation $R(X, Y)Z = \langle L(Y), Z \rangle L(X) - \langle L(X), Z \rangle L(Y)\)$

Solution.

(a) This symmetry is an immediate consequence of the definition of $R$:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = - R(Y, X).$$
(b) First observe:

\[ XY(Z, W) = X(\nabla_Y Z, W) + X(Z, \nabla_Y W) \]
\[ = (\nabla_X \nabla_Y Z, W) + (\nabla_Y Z, \nabla_X W) + (\nabla_X Z, \nabla_Y W) + (Z, \nabla_X \nabla_Y W). \]

It follows that

\[ [X, Y](Z, W) = (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, W) - (Z, \nabla_X \nabla_Y W - \nabla_Y \nabla_X W). \]

On the other hand

\[ [X, Y](Z, W) = (\nabla_{[X,Y]} Z, W) + (Z, \nabla_{[X,Y]} W). \]

Subtracting the two previous equalities gives

\[ 0 = \langle R(X, Y) Z, W \rangle + \langle Z, R(X, Y) W \rangle, \]

which is what we wanted to show.

(c) Using the torsion-free property:

\[
R(X, Y) Z + R(Y, Z) X + R(Z, X) Y = (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) + (\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X) \\
+ (\nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y) \\
= \nabla_X [Y, Z] + \nabla_Y [Z, X] + \nabla_Z [X, Y] - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\
= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\
= 0.
\]

(d) Note that if \( B \) is a bilinear, skew symmetric 2-form on \( T_p M \) then

\[
B(f_1, f_2) = B(a_{11} e_1 + a_{21} e_2, a_{12} e_1 + a_{22} e_2) = (a_{11} a_{22} - a_{21} a_{12}) B(e_1, e_2) = \det(A) B(e_1, e_2).
\]

Clearly, we also have \( B(f_2, f_1) = (\det A) B(e_2, e_1) \). Combining this observation and the symmetries of \( R \) of parts (a) and (b) gives

\[
\langle R_p(f_1, f_2) f_2, f_1 \rangle = (\det A)^2 \langle R_p(e_1, e_2) e_2, e_1 \rangle.
\]

If the two bases are orthonormal, \( (\det A)^2 = 1 \), so the quantity \( K_p(V) \) does not depend on the choice of orthonormal basis of \( V \).

(e) Recall that for the sphere of radius \( R \), we have \( L(u) = -\frac{1}{R} u \) for all \( u \in p \) and any \( p \) on the sphere. If \( e_1, e_2 \) are any two orthonormal vectors tangent to the sphere at a given point \( p \) and \( V \) is the linear span of these two vectors, then

\[
K_p(V) = \langle L(e_2), e_2 \rangle \langle L(e_1), e_1 \rangle - \langle L(e_1), e_2 \rangle \langle L(e_2), e_1 \rangle = \frac{1}{R^2}.
\]

\[ \diamond \]

3. **Geometry of tubes in \( \mathbb{R}^3 \).** Let \( c(s) \) be a smooth closed curve in \( \mathbb{R}^2 \) parametrized by arc-length, where \( \mathbb{R}^2 \) is regarded as a plane in \( \mathbb{R}^3 \). Let \( e_1(s) := c'(s) \) (the tangent vector to the curve) and \( e_2(s) \) the unit orthogonal vector to \( e_1(s) \) so that \( e_3 := e_1(s) \times e_2(s) \) is the standard basis vector \( (0, 0, 1) \) of \( \mathbb{R}^3 \). Define the parametric surface \( M \subseteq \mathbb{R}^3 \) with parametrization

\[
\Phi(s, \varphi) = c(s) + r \left[ \cos(\varphi) e_2(s) + \sin(\varphi) e_3 \right]
\]
for $0 \leq \varphi \leq 2\pi$. Notice that the image of $\Phi$ is a tube of radius $r$ with central (plane) curve $c(s)$. Let $\kappa(s)$ denote the (signed) curvature of $c$ (textbook, page 11), so that

$$e_1'(s) = \kappa(s)e_2(s), \quad e_2'(s) = -\kappa(s)e_1(s).$$

We further define the vector fields $E_1, E_2$ on the parametric surface $M$ by

$$E_1(s, \varphi) = \frac{\partial \Phi}{\partial s}(s, \varphi), \quad E_2(s, \varphi) = \frac{\partial \Phi}{\partial \varphi}(s, \varphi), \quad N(s, \varphi) = \cos(\varphi)e_2(s) + \sin(\varphi)e_3.$$

(a) Show that $E_1, E_2$ constitute an orthonormal frame on $TM$ (i.e., an orthonormal basis of $T_pM$ at each point $p = \Phi(s, \varphi)$, and that $N$ is a unit normal vector field to $M$.

(b) Show that the shape operator $L$ of the tube satisfies at each point of $M$:

$$L(E_1(s, \varphi)) = -\frac{\kappa(s)\cos(\varphi)}{1 - r\kappa(s)\cos(\varphi)}E_1(s, \varphi), \quad L(E_2(s, \varphi)) = \frac{1}{r}E_2(s, \varphi).$$

(c) What is the Gaussian curvature of the parametric surface at each point $p = \Phi(s, \varphi)$?

(d) Let $\nabla$ denote the Levi-Civita connection on $M$. Obtain the vector fields:

$$\nabla_{E_1}E_1, \quad \nabla_{E_2}E_2, \quad \nabla_{E_1}E_2, \quad \nabla_{E_2}E_1.$$

Solution.

(a) A calculation gives

$$E_1(s, \varphi) = e_1(s), \quad E_2(s, \varphi) = -\sin(\varphi)e_2(s) + \cos(\varphi)e_3, \quad N(s, \varphi) = \cos(\varphi)e_2(s) + \sin(\varphi)e_3.$$

Taking dot products we see that these three vectors form an orthonormal basis of $\mathbb{R}^3$ at each point on the surface. In particular, $N$ is perpendicular to the surface and outward pointing.

(b) Note that

$$D_{E_1}N = -\frac{1}{\|\partial \Phi / \partial s\|} \frac{\partial N}{\partial s}(s, \varphi) = -\frac{\kappa(s)\cos(\varphi)}{1 - r\kappa(s)\cos(\varphi)}E_1(s, \varphi)$$

and

$$D_{E_1}N = \frac{1}{\|\partial \Phi / \partial \varphi\|} \frac{\partial N}{\partial \varphi}(s, \varphi) = \frac{1}{r}[-\sin(\varphi)e_2(s) + \cos(\varphi)e_3] = \frac{1}{r}E_2(s, \varphi).$$

Taking the negative of these expressions we obtain the values of the shape operator applied to $E_1, E_2$.

(c) The above give the principal curvatures

$$\kappa_1 = \frac{\kappa(s)\cos(\varphi)}{1 - r\kappa(s)\cos(\varphi)}, \quad \kappa_2 = -\frac{1}{r}.$$

The Gaussian curvature of the surface is the product of the principal curvatures:

$$K(s, \varphi) = -\frac{1}{r} \frac{\kappa(s)\cos(\varphi)}{1 - r\kappa(s)\cos(\varphi)}.$$
(d) Denoting by $\Pi$ the orthogonal projection to the tangent space to $M$ at the point $p = \Phi(s, \varphi)$

\[
\nabla E_1 E_2 = \frac{1}{\|\frac{\partial \Phi}{\partial s}\|} \Pi \frac{\partial}{\partial s} [-\sin(\varphi)e_2(s) + \cos(\varphi)e_3] \\
= \frac{1}{1 - r\kappa(s) \cos(\varphi)} \Pi [-\sin(\varphi)e'_2(s)] \\
= \frac{1}{1 - r\kappa(s) \cos(\varphi)} \Pi [\sin(\varphi)\kappa(s)e_1(s)] \\
= \frac{\kappa(s) \sin(\varphi)}{1 - r\kappa(s) \cos(\varphi)} E_1.
\]

We also have

\[
\nabla E_2 E_2 = \frac{1}{\|\frac{\partial \Phi}{\partial \varphi}\|} \Pi \frac{\partial}{\partial \varphi} [-\sin(\varphi)e_2(s) + \cos(\varphi)e_3] \\
= \frac{1}{r} \Pi [-\cos(\varphi)e_2(s) - \sin(\varphi)e_3] \\
= -\frac{1}{r} \Pi N(s, \varphi) \\
= 0.
\]

Now observe that

\[
\nabla E_2 E_1 = \langle \nabla E_2 E_1, E_1 \rangle E_1 + \langle \nabla E_2 E_1, E_2 \rangle E_2.
\]

The first inner product on the right-hand side is 0 is $E_1$ has constant norm (equal to 1). The second inner product is also zero since $\langle \nabla E_2 E_1, E_2 \rangle = -\langle E_1, \nabla E_2 E_2 \rangle = 0$. Therefore $\nabla E_2 E_1 = 0$. A similar argument gives

\[
\nabla E_1 E_1 = -\langle \nabla E_1 E_2, E_1 \rangle E_2 = -\frac{\kappa(s) \sin(\varphi)}{1 - r\kappa(s) \cos(\varphi)} E_2.
\]

Thus we obtain all the covariant derivatives:

\[
\nabla E_1 E_1 = -\frac{\kappa(s) \sin(\varphi)}{1 - r\kappa(s) \cos(\varphi)} E_2, \quad \nabla E_2 E_1 = 0, \quad \nabla E_1 E_2 = 0, \quad \nabla E_2 E_2 = 0.
\]

4. **The gradient and Hessian of a function.** (20 points.) Let $M$ be a Riemannian manifold and $f : M \to \mathbb{R}$ a smooth function. We define the *gradient* of $f$ as the vector field $\text{grad} f$ such that, for every other vector field $Z$, $p \in M$ and $v \in T_p M$,

\[
\langle (\text{grad} f)(p), v \rangle = vf.
\]

Note that $vf = df(v)$ is the directional derivative of $f$ along $v$. We define the *Hessian* of $f$ at $p$ as the quadratic form on $T_p M$ given by

\[
\text{Hess}(f)_p(u, v) := \langle \nabla_u \text{grad} f, v \rangle
\]

where $\nabla$ is the Levi-Civita connection on $M$. Show that the Hessian is a symmetric quadratic form: $\text{Hess}_p(u, v) = \text{Hess}_p(v, u)$. (Note: we may define the *Laplacian* of the Riemannian manifold $M$ as the trace of the Hessian, $\text{Tr}(\text{Hess}(f))$.)
Solution. Let $U, V$ be smooth vector fields on $M$ such that $U_p = u, V_p = v$. Note that

$$\langle \nabla_U \text{grad} f, V \rangle = U \langle \text{grad} f, V \rangle - \langle \text{grad} f, \nabla_U V \rangle.$$ 

Therefore

$$\text{Hess}(f)(U, V) - \text{Hess}(f)(V, U) = U \langle \text{grad} f, V \rangle - \langle \text{grad} f, \nabla_U V \rangle - \langle V \langle \text{grad} f, U \rangle - \langle \text{grad} f, \nabla_V U \rangle \rangle$$

$$= UV f - VU f - \langle \text{grad} f, [U, V] \rangle$$

$$= 0.$$